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AN ANALYSIS OF THE LEARNING-UNITS IN N PROCESSES IN ALGEBRA

By GLENN R. PEASE

Stockton, Calif.

THE LEARNING-UNIT CONCEPT

In the face of continued agitation for curricular revision, looking toward the "Practical" in all lines of secondary school subjects, it may, at first thought, seem a bit archaic to propose a study of the purely abstract elements involved in the fundamental functions of beginning algebra. Such a study is desirable, however, because the mastery of these abstract principles is an absolute prerequisite to successful problem solving in the field which the curriculum revisors point out as proper material for first year algebra. Such necessity for mastery of the abstract problem preliminary to verbal problem solving is recognized by Judd,[1] Rugg and Clark,[2] and other investigators. H. E. Dow[3] points out the fact that in nine of the most widely used textbooks, 80 percent of the problems offered are abstract problems, and only 20 percent are verbal problems.

The author's study approaches the question from the psychological viewpoint of the learning-unit, or unit-skill. Although well aware of the many logical, psychological, and even physiological difficulties involved in Thorndike's situation-response bond theory, that concept seems to offer the most practical classroom approach to the learning problem. A learning-unit is defined, therefore, as any situation-response bond, or hierarchy

[1] Judd, C. H., *The Psychology of High School Subjects*, p. 105, Ginn & Co., Boston, 1915.

[2] Rugg, H. O. and Clark, J. R., *Standardized Tests and the Improvement of Teaching in First Year Algebra.* School Review, XXV, p. 116.

[3] Dow, H. E., *The Distribution of Abstract and Verbal Problems in Algebra*, Iowa M.A. Thesis, 1926, p. 49.

of such bonds, which operates as a psychological unit. Such learning-units are not fixed and arbitrary things—they are functioning units. Thus, what may be at an early stage of the learning process, an extended series of more or less independent units, often becomes fused, at a later stage, into a single unit.[4]

For illustration, take a problem in the addition of directed numbers. In $(+4) + (-4) = 0$ the figure "4" represents a learning-unit carried over directly from arithmetic, as do also the "equals" sign and the "plus" sign between the parentheses. The combination of the + and the 4 in the first parentheses represents a new concept, or a new bond for the algebra pupil. It involves the + sign, not as a sign of function as in arithmetic, but as a sign of direction from the zero point. The $+4$ then becomes a learning-unit. Similarly -4 is a learning-unit involving the arithmetic concept of "fourness" and the added idea of the negative sign as a sign of direction from zero. Each of these units is thus seen to be a hierarchy of bonds. But the $+4$ and the -4 are not only learning-units in themselves; they also combine into a more complex hierarchy in the learning-unit, $(+4) + (-4) = 0$. This is at first a difficult concept for the pupil, but with sufficient drill it becomes as simple as the better known learning-unit from the arithmetic field, $4 - 4 = 0$.

But not only must $(+4) + (-4) = 0$ function as a learning unit; the very idea of "plusness" and "minusness" of numbers must become dissociated until it will attach to any similar set of equal numbers, such as $(+3)+(-3)=0$ or $(+13)+(-13)=0$. There is a clear recognition that the problem involving 13 may be more difficult than that involving 3, but for the sake of getting into the algebra research, we assume that arithmetic learning-units are already mastered.[5]

In like manner $(+a) + (-a) = 0$ is a learning unit which must be varied to $(+b) + (-b) = 0$, to $(+ab) + (-ab) = 0$, and so on until the plusness and minusness are dissociated, and generalized to the point that equality of literal terms following them in addition will automatically call forth the response of "zero."

An algebraic ability is then but the summation of the skills, drilled to the point of automatic response, in some pedagogical

[4] Perhaps "procedure-unit" or "operation-unit" would now be a more exact term than "learning-unit."

[5] As a matter of fact the completed study shows that out of nearly 44,000 errors made, 4.7 percent were arithmetic errors.

division of the total algebra field. For example, we may say that Jane Hewitt has fair ability in algebraic addition of known numbers, while Sam Brown has average ability, and Bessie Polk has excellent ability in the same field. We actually mean that Jane Hewitt has only a fair sum of algebraic learning-units, such as $+4+(-4)=0$ and $-4+(+7)=+3$ which have been drilled upon until they function automatically. Sam Brown has about as many of such automatically functioning units as the average of the class, while Bessie Polk has most of the possible learning-units involved in addition of algebraic quantities functioning in an automatic manner with a high degree of accuracy, say 99 times out of 100.

Likewise the degree of mastery of the subject of first year algebra is but the summation of the multitudinous learning-units in their hierarchy forms, covering all the subdivisions of the total field.

From the viewpoint of such an approach it is not surprising that effort to measure algebraic ability by a sort of hit-or-miss selection of problems has given only partial success. How, in fact, can we expect to properly drill, test, or study abilities until we have definitely determined what are the bonds or learning-units involved?

Many of the investigators [6] in the field have stressed the desirability of such a scientific analysis of the learning-units involved in beginning algebra. E. L. Thorndike clearly points out the fundamental nature of the learning-unit in algebra in the following words: "Learning algebraic computation is and should be, in large measure, the formation and organization of a hierarchy of mental connections or bonds. The science of teaching algebra must consider what these bonds are, the amount of practice each should have, how this practice should be distributed, how the order and methods of formation of these bonds may provide the maximum of facilitation and the minimum of interference among them." [7]

[6] Samples of such expression of dissatisfaction can be found in the following: Symonds, P. M., *The Psychology of Errors in Algebra*, MATHEMATICS TEACHER, XV, p. 102. Fossler, M. L., *Errors Made by Students Who have Completed First Year Algebra, as Shown by the Douglas Standard Tests for Elementary Algebra*, unpublished Iowa M.A. thesis, 1924. Palmer, E. L., *The Analysis of Algebra in Relation to the Learning Process*, Iowa M.A. Thesis, 1925, p. 180.

[7] Thorndike, E. L., *The Psychology of Algebra*, p. 250, The Macmillan Co., New York, 1925.

The Problem

Having accepted the learning-unit as the functional unit of algebraic skill, the problem is clearly four-fold. First, identification of those individual unit-skills or "S-R bonds" that, collectively considered, make up the subject matter of the fundamental processes in first year algebra. Second, the determination of the relative difficulty of the above identified learning-units. Third, identification of the typical errors connected with the mastery of each learning-unit, or pedagogical group of such units; and the determination of the relative proportions of their occurrence. Fourth, how much teaching, how many examples, and how many drills should be used in order to bring, in the most economical manner, each learning-unit or group of units to the point where they function correctly and more or less automatically for the average pupil?

The identification of the learning-units is obviously the initial problem. Taking for granted the automatic functioning of all arithmetic skills involved, the total possible variety of units is so great that one is inclined to group slightly different skills under a common head. For example such questions as the following confront the one who undertakes such an analysis: will the learning of such a unit as $a - a = 0$ transfer to $ab - ab = 0$, and to $(a - b) - (a - b) = 0$? Or should we consider each of the above as a learning-unit in itself? One can make any arbitrary choice among various lines of sub-grouping, and then declare, as some investigators have done, that there are so many learning-units in the field. But the only scientific way to approach the problem, is to teach one and test for transfer to the others. The final identification of the learning-units is then not a matter of individual taste or inclination, but of discovery through a test program covering all possible units.

The relative difficulty of the various learning-units can be readily determined by the study of the above test results, remembering always that such testing must have taken place immediately after the completion of the pedagogical subdivision of the subject matter, for such testing programs as that conducted by Rugg [8] with pupils "just finishing first year algebra" will obviously distort the findings in respect to both relative difficulty of the

[8] Rugg, H. O., *The Experimental Determination of Standards in First Year Algebra*. School Review, XXIV, pp. 37–66, Jan., 1916.

earlier learned units, and also the relative frequency of errors incident to the learning of such units.

Typical errors can be determined by an analytical study of the problems with wrong answers in the pupils' test papers; and the relative frequency of each type of error easily tabulated and totaled.

The problem of the amount of teaching, the number of examples, and the amounts of drill required for the mastery of each learning-unit demands more complete control of experimental schools than was possible in the present study. An approach to the problem was made, however, by analyzing the various textbooks used in the schools cooperating in the research. The technique necessary to the study of such a problem depends very largely on the findings of the three earlier problems and the research is therefore left for later investigation.

General Plan and Procedure

In making the analysis of possible learning-units, the author carefully studied several textbooks and drill-books, noting the various types of skills which were given clean-cut and specific attention. With these skills as a beginning, he varied the elements, and added such new units as seemed necessary to make a complete survey of the field.

Tests were then constructed for the various pedagogical divisions of the field, each test including representative problems for each of the above identified learning-units. In most cases two or three problems of each kind were included in order to raise the reliability of the tests.[9] Test material was printed with all instructions necessary for the pupil's understanding of the test procedure. Tests were held by the principal of the experimental school until the textbook material covering the particular field had been covered, thus avoiding all chance of teachers drilling pupils on the exact units of the test program. Instructors were cautioned to teach only such material as was found in the text, and to introduce no drill problems from outside the text.

At the completion of any unit of textbook material, the corresponding test was given. Test results for all "repeaters"

[9] Reliability, determined by correlation between halves, and use of Brown's Formula, shows a range between .83 and .95 with an A.M. of .90. See Table XXXVIII.

were eliminated from the data. Tests were corrected by carefully chosen clerical help using stencil correction-forms prepared by the author.

Four representative high schools cooperated in the test program: Albia, Iowa, with Newell and Harper's *A Year of Algebra* as a text; Oskaloosa, Iowa, with Wells and Hart's *Modern First Year Algebra* as a text; Marshall, Mo., with Schorling and Clark's *Modern Algebra, Ninth School Year* as a text; Charles City, Iowa, which acted as alternate for Marshall, and completed the latter half of the test program. All of these cities are small enough to contain a large contingent of rural pupils. Each of them is in a small way a manufacturing community. Two of them are railroad division points, and two are in mining sections. One city has a small college in it, another has two small colleges. They give a total population of over 350 first year algebra pupils. Test data from all pupils, except "repeaters," are included in the study, thus giving a good sampling of middle west high school population.

Treatment of Data

Relative difficulty of the learning-units in any field,[10] such as vertical addition of known numbers, was found by reducing the raw data of wrongs for each particular learning-unit to wrongs per one hundred pupils for one representative problem. Such measures are now comparable both within the pedagogical division covered by the test, and throughout the study as a whole. Any problem was counted as wrong when it contained an error, or when it was omitted in the body of the test. Problems omitted by very slow pupils at the ends of the tests were not counted as wrongs.

In each pedagogical division of the algebra field covered by the research, an analytical study of wrong problems was made to determine the types of errors made, and the relative frequency of each type of error. Where a type of error occurred only once or twice in the test data of the total population it was listed as an unclassified error. Where the wrong answer might have been arrived at by two or more different types of error, it was listed as unclassified; where it was impossible to determine from the pupil's work just how he arrived at his wrong answer, the

[10] Relative difficulty of the learning-units is presented in Tables I to XVII. Relative frequency of type errors, and the contribution of each type to total difficulty in the particular field are found in Tables XVIII to XXXVII.

error was listed as unclassified. It was thus possible to determine the percent contributed by any particular type of error to the total difficulty in any field; and finally to determine the constituent elements and their proportionate contributions to total algebraic difficulty in the whole field covered by the research.

The three textbooks were analyzed to determine the amount of teaching, the number of examples, and the number of drill problems offered for each of the learning-units identified. The wrongs per one hundred pupils for each experimental school per one representative problem were now correlated with the drill problems offered; also the number of errors made per one hundred pupils in each experimental school for each division of the field was correlated with the number of drill problems offered. The value of r between "wrongs" and "drill problems" was $+.13$; between "errors" and "drill problems" the value of r was $+.08$. Apparently, with our incomplete control of the experimental schools, the "teacher equation" completely over-topped the matter of drill problems offered by the text.

Significance of Findings

With the large number of cases, drawn from four separate schools using three different textbooks, our data should be rather indicative of the relative difficulty of the learning-units included in the test material. Unfortunately, due to typographical errors in the printing, and in one case to cutting off of several problems at the end of the test, there are some learning-units with no respresentation in the tests, and therefore no data.

The author makes no claim that the findings in respect to relative difficulty should be accepted as absolute. Where the wrongs per one hundred pupils are 0 or only 1 or 2, it seems clear that the textbook offerings and teaching were sufficient for the pupils' learning; perhaps time was wasted in some cases by over-learning due to too many drill problems. But where quite large numbers fail per one hundred pupils, it is clear that we have either a very difficult learning-unit, or very inadequate treatment by the textbook. One very significant case is that of zero times a number, where approximately twenty-five percent of the pupils fail. None of the three texts offers a single drill problem for this type of learning-unit. In general, the relative difficulty as stated in the tables gives a quite adequate idea of where

emphasis should be put in teaching. Later investigators with more perfect control of drill and teaching procedure may find it possible to make a more accurate measurement of relative difficulty.

TABLE I

RELATIVE DIFFICULTY OF LEARNING-UNITS IN ADDITION OF KNOWN NUMBERS

Like Signs	Augend[1] > Addend			Augend < Addend			Augend = Addend		
	Ex.	Rel. Dif.		Ex.	Rel. Dif.		Ex.	Rel. Dif.	
		V.	H.		V.	H.		V.	H.
"+" sign printed.	+9 +4	1	0	+4 +9	1	0	+4 +4	1	0
One "+" printed.	+9 4	17	7	+4 9	4	6	+4 4	24	*
Negative signs.	−9 −4	8	9	−4 −9	8	9	−4 −4	*	*
Unlike Signs									
Augend "+".	+9 −4	2	3	+4 −9	6	5	+4 −4	3	1
"+" omitted.	9 −4	4	4	4 −9	6	4	4 −4	*	*
Augend "−".	−9 +4	5	5	−4 +9	6	5	−4 +4	3	3
Zero units.	+4 0	2	0	0 +4	1	1			
	0 −4	4	2	−4 0	2	2			
2 Neg. and 1 Pos.		3	3						
2 Pos. and 1 Neg.		3	2						
More than 3 No.'s mixed Sn.		4	5						

[1] *Augend* is the number to be increased.

Legend: Ex., example of the learning-unit indicated. Rel. Dif., relative difficulty in terms of wrongs per one hundred pupils per representative problem in the test material. V., wrongs per one hundred pupils per problem for vertical form. H., wrongs per one hundred pupils per problem for horizontal form.

* Signs changed in printing so that no representative problem occurred in the tests, therefore no data on this learning-unit.

TABLE II. Relative Difficulty of Learning-units in Addition of Literal Numbers

	Augend > Addend						Augend < Addend						Augend = Addend					
	1 Pr. Coef.			2 Pr. Coef.			1 Pr. Coef.			2 Pr. Coef.			2 Pr. Coef.			No. Pr. Coef.		
	Ex.	V.	H.	Ex.	V.	H.	Ex.	V.	H.	Ex.	V.	H.	Ex.	V.	H.	Ex.	V.	H.
Like signs "+" signs	$+4a$ $+a$	10	5	$+4a$ $+2a$	3	2	$+a$ $+4a$	10	7	$+2a$ $+4a$	3	3	$+4a$ $+4a$	3	3	$+a$ $+a$	11	5
"−" signs	$-4a$ $-a$	14	12	$-4a$ $-2a$	4	7	$-a$ $-4a$	13	8	$-2a$ $-4a$	4	*	$-4a$ $-4a$	*	*	$-a$ $-a$	23	13
Unlike signs Augend "+"	$+4a$ $-a$	13	8	$+4a$ $-2a$	4	3	$+a$ $-4a$	12	10	$+2a$ $-4a$	4	5	$+4a$ $-4a$	3	2	$+a$ $-a$	7	2
Augend "−"	$-4a$ $+a$	19	20	$-4a$ $+2a$	4	5	$-a$ $+4a$	20	17	$-2a$ $+4a$	4	4	$-4a$ $+4a$	4	3	$-a$ $+a$	8	4
Zero units "+" No.	$+4a$ 0	or		$+a$ 0	3	1	0 $+4a$	or		0 $+a$	4	3						
"−" No.	0 $-4a$	or		0 $-a$	4	4	$-4a$ 0	or		$-a$ 0	4	3						
Two Pos. numbers and one Neg. number										—	3	4						
Two Neg. numbers and one Pos. number											4	4						
More than three numbers, mixed signs											11	9						

Legend: Pr. Coef., is abbreviation for printed coefficient. Ex., example of learning-unit indicated. Relative difficulty of the unit is expressed in terms of wrongs per one hundred pupils per one representative problem in test material. V., relative difficulty for vertical form. H., relative difficulty for horizontal form.

* Signs changed in printing so that no representative problem occurred in the tests, therefore no data on this learning-unit.

Note: Learning-units where one of the "+" signs is omitted were not included in this table because (1) it was thought that the meaning of a number without a sign was already learned by the average pupil by this time, (2) a few cases of omitted sign problems in the test material prove the statement of (1) to be correct for our schools.

Note: In the test material problems were varied in literal content using *a*, *ab*, *abc*, forms with varied letters.

TABLE III

Relative Difficulty of Learning-units in Addition of Polynomials

Learning-unit	Examples				Rel.Dif.
Binom.—One Lit. and one known term.					
Sum and difference	$\begin{array}{r}4a+2\\ 9a-5\\ \hline\end{array}$	$\begin{array}{r}4a+2\\ -9a-5\\ \hline\end{array}$	$\begin{array}{r}4a-2\\ 9a+5\\ \hline\end{array}$	etc.	8
Difference and difference	$\begin{array}{r}4a-2\\ 9a-5\\ \hline\end{array}$	$\begin{array}{r}4a-2\\ -9a-5\\ \hline\end{array}$		etc.	6
Both terms literals					
No numerical coefficients printed	$\begin{array}{r}a+b\\ a-b\\ \hline\end{array}$	$\begin{array}{r}a-b\\ -a-b\\ \hline\end{array}$		etc.	10
Printed Num.Coef. in augend or addend only					
Sum and difference	$\begin{array}{r}a+b\\ 4a-3b\\ \hline\end{array}$	$\begin{array}{r}-4a-3b\\ a+b\\ \hline\end{array}$		etc.	6
Difference and difference	$\begin{array}{r}a-b\\ 4a-3b\\ \hline\end{array}$	$\begin{array}{r}-4a-3b\\ a-b\\ \hline\end{array}$		etc.	8
Num.Coef. in both augend and addend					
Sum and difference	$\begin{array}{r}2a+5b\\ 4a-3b\\ \hline\end{array}$	$\begin{array}{r}-2a-5b\\ 4a+3b\\ \hline\end{array}$		etc.	6
Difference and difference	$\begin{array}{r}2a-5b\\ 4a-3b\\ \hline\end{array}$	$\begin{array}{r}-4a-3b\\ 2a-5b\\ \hline\end{array}$		etc.	6
Monomial and binomial					
Monomial like first term		$\begin{array}{l}\ \ 4a-3b\\ -2a\\ \hline\end{array}$		etc.	5
Monomial like second term	$\begin{array}{r}4a-3b\\ +5b\\ \hline\end{array}$	$\begin{array}{r}-3b\\ -2a+5b\\ \hline\end{array}$		etc.	6
Monomial and two or more binomials		$\begin{array}{l}\ \ 4a-3b\\ -9a\\ \ \ 2a+5b\\ \hline\end{array}$		etc.	6
Binomial and trinomial	$\begin{array}{rr}2a+5b & -3c\\ -4a & +2c\\ \hline\end{array}$	$\begin{array}{r}2a+5b-3c\\ -3b+7c\\ \hline\end{array}$		etc.	11
Three or more polynomials with scattered terms	$\begin{array}{l}4a-3b\qquad -7d\\ 2a-5b+2c\\ \qquad 6b-4c+3d\\ \hline\end{array}$				26
Rearrange terms and add, trinomials or greater					16

Legend: Relative difficulty is expressed in terms of wrongs per one hundred pupils per one representative problem in the test material.

TABLE IV

RELATIVE DIFFICULTY OF LEARNING-UNITS IN SUBTRACTION OF KNOWN NUMBERS

	Min. > Subtr.			Min. < Subtr.			Min. = Subtr.		
	Ex.	Rel. Dif.		Ex.	Rel. Dif.		Ex.	Rel. Dif.	
		V.	H.		V.	H.		V.	H.
Like signs									
+ from +	+9 +4	4	4	+4 +9	8	8	+4 +4	3	2
One sign omitted	+9 4	7	27	+4 9	8	13	+4 4	4	28
− from −	−9 −4	6	7	−4 −9	10	11	−4 −4	2	3
Unlike signs									
− from +	+9 −4	7	9	+4 −9	8	18	+4 −4	6	16
+ from −	−9 +4	12	11	−4 +9	10	*	−4 +4	12	11
Involving zero	+4 0	5	3	−4 0	5	4			
	0 −4	11	9	0 +4	13	11			
Double subtraction, horizontal form only						18			

Legend: Rel.Dif., relative difficulty of unit is expressed in terms of wrongs per one hundred pupils per one representative problem in the test material. V., relative difficulty of vertical form. H., relative difficulty of horizontal form. Ex., example of learning-unit indicated.

* Signs changed in printing so that there was no representative problem in the tests, therefore no data for this unit.

TABLE V

RELATIVE DIFFICULTY OF LEARNING-UNITS IN SUBTRACTION OF KNOWN NUMBERS, INVERTED FORM

	Difference, +			Difference, −			Difference, zero		
	Ex.	V.	H.	Ex.	V.	H.	Ex.	V.	H.
Like signs									
+ from + ...	+4			+13			+4		
	+13	3	3	+ 4	6	7	+4	4	3
− from −	−13			− 4			−4		
	− 4	6	7	−13	6	6	−4	3	3
Unlike signs									
− from + ...	− 4								
	+ 9	12	10						
+ from −				+4					
				−9	10	9			
Zero units									
9 from + or −	0			0					
	+4	4	3	−4	6	5			
+ or − from 0	−4			+4					
	0	6	6	0	9	6			

Legend: V., relative difficulty in terms of wrongs per one hundred pupils per one representative problem in test material, vertical form. H., relative difficulty in terms of wrongs per one hundred pupils per one representative problem in test material, horizontal form. Ex., example of learning-unit indicated.

TABLE VI. Relative Difficulty of Learning-units in Subtraction of Literal Numbers

	Min. > Subtr.			Min. < Subtr.			Min. = Subtr.		
	Ex.	V.	H.	Ex.	V.	H.	Ex.	V.	H.
Like signs									
+ from +									
No Num. Coef. . . .							*a* *a*	*	*
One Num. Coef. . . .	4*a* *a*	9	11	*a* 4*a*	11	13			
Two Num. Coef. . .	9*a* 4*a*	4	5	4*a* 9*a*	8	10	4*a* 4*a*	6	12
− from −									
No Num. Coef. . . .							−*a* −*a*	*	*
One Num. Coef. . . .	−4*a* −*a*	8	12	−*a* −4*a*	8	8			
Two Num. Coef. . .	−9*a* −4*a*	6	7	−4*a* −9*a*	7	9	−4*a* −4*a*	4	5
Unlike signs							−*a* *a* *a* −*a*	*	*
One Num. Coef.									
− from + ...	4*a* −*a*	8	13	*a* −4*a*	12	11			
+ from −	−4*a* *a*	17	11	−*a* 4*a*	*	22			
Two Num. Coef.									
− from + ...	9*a* −4*a*	6	7	4*a* −9*a*	8	10	4*a* −4*a*	6	9
+ from −	−9*a* 4*a*	19	11	−4*a* 9*a*	13	21	−4*a* 4*a*	* 11	19
Zero units									
No Num. Coef.	*a* 0	2	4	−*a* 0	2	4			
	0 −*a*	6	9	0 +*a*	13	26			
Num. Coef.	4*a* 0	2	8	−4*a* 0	4	10			
	0 −4*a*	6	8	0 4*a*	7	24			

Double subtraction, horizontal form: All + 28 All − 18
Mixed signs 26

Legend: V., relative difficulty in terms of wrongs per one hundred pupils per one representative problem in test material. H., relative difficulty in terms of wrongs per one hundred pupils per one representative problem in test material. Ex., example of learning-unit indicated.

* Signs changed in printing or problem omitted in test material.

TABLE VII

Relative Difficulty of Learning-units in Subtraction of Polynomials

					Relative Difficulty
Sum from sum	+ +	+ +	− +	− +	
+	+ +	− +	+ +	− +	9
Difference from sum	+ +	+ +	− +	− +	
	+ −	− −	+ −	− −	9
Sum from difference	+ −	+ −	− −	− −	
	+ +	− +	+ +	− +	12
Difference from dif.	+ −	+ −	− −	− −	
	+ −	− −	+ −	− −	9

Monomial from trinomial	6
Trinomial from monomial	15
Binomial from trinomial	10
Trinomial from binomial	19
Trinomial from trinomial	13
Trinomial from trinomial, two terms common	18
Binomial from binomial, one term common	18
Rearrange terms and subtract	29

Legend: Relative difficulty in terms of wrongs per one hundred pupils per one representative problem.

TABLE VIII

Relative Difficulty of Learning-units in Multiplication of Known Numbers

	Pos. No.			Pos. 1			Neg. No.			Neg. 1			Zero.		
		V.	H.		V.	H.		V.	H.		V.	H.		V.	H.
Pos. times	+4 +9	3	1	+1 +9	1	1	−4 +9	1	2	−1 +5	1	1	0 +5	21	21
Neg. times	+4 −9	1	2	+1 −9	1	1	−4 −9	3	3	−1 −9	2	2	0 −5	23	21
Pos. 1 times	+4 +1	1	1	+1 +1	*	*	−4 +1	1	1	−1 +1	*	*	0 +1	*	*
Neg. 1 times	+4 −1	1	1	+1 −1	*	*	−4 −1	2	1	−1 −1	*	*	0 −1	*	*
Zero times	+4 0	24	23	+1 0	*	*	−4 0	25	19	−1 0	*	*			

Horizontal form, double multiplication:	
(+)(+)(+) =	2
(−)(−)(−) =	12
(+)(+)(−) =	4
(−)(−)(+) =	11
(−)(−)(−)(−) =	3
(−)(−)(−)(+) =	7

Legend: V., relative difficulty in terms of wrongs per one hundred pupils per one representative problem, vertical form. H., relative difficulty in terms of wrongs per one hundred pupils per one representative problem, horizontal form.

* Representative problems for these learning-units omitted from the test by mistake, therefore no data.

TABLE IX. Relative Difficulty of Learning-units in Multiplication of Literal Numbers

Multiplier	Without Numerical Coefficients				With Numerical Coefficients				Double Mult.			
No Exponents	Positive	D.	Negative	D.	Positive	D.	Negative	D.	No. Exponents	D.	Exponents	
Positive	$(a)(b) =$	2	$(a)(-b) =$	11	$(a)(2b) =$	3	$(a)(-2b) =$	12	$(a)(b)(c) =$	6	$(a)(ab)(ac) =$	18
					$(2a)(3b) =$	8	$(2a)(-3b) =$	8	$(a)(b)(-c) =$	10	$(a)(ab)(-ac) =$	15
Negative	$(-a)(b) =$	3	$(-a)(-b) =$	5	$(-a)(2b) =$	4	$(-a)(-2b) =$	9	$(a)(-b)(-c) =$	10	$(a)(-ab)(-ac) =$	13
					$(-2a)(3b) =$	7	$(-2a)(-3b) =$	9	$(-a)(-b)(-c) =$	12	$(-a)(-ab)(-ac) =$	17
Exponents									$(2a)(3b)(5c) =$	6	$(2a)(3ab)(5ac) =$	19
Positive	$(a)(a) =$	14	$(a)(-a) =$	18	$(a)(2a) =$	13	$(a)(-2a) =$	14	$(2a)(3b)(-5c) =$	11	$(2a)(3ab)(-5ac) =$	23
	$(a)(ab) =$	9	$(a)(-ab) =$	11	$(a)(2ab) =$	8	$(a)(-2ab) =$	12	$(2a)(-3b)(-5c) =$	12	$(2a)(-3ab)(-5ac) =$	13
					$(2a)(3ab) =$	9	$(2a)(-3ab) =$	11	$(-2a)(-3b)(-5c) =$	14	$(-2a)(-3ab)(-5ac) =$	15
Negative	$(-a)(a) =$	16	$(-a)(-a) =$	15	$(-a)(3a) =$	13	$(-a)(-3a) =$	14				
	$(-a)(ab) =$	12	$(-a)(-ab) =$	13	$(-a)(3ab) =$	10	$(-a)(-3ab) =$	13				
					$(-2a)(3ab) =$	9	$(-2a)(-3ab) =$	11				

Legend: D., relative difficulty expressed in terms of wrongs per one hundred pupils per one representative problem.

TABLE X. Relative Difficulty of Learning-units in Multiplication of Polynomials by a Monomial

	No Exponents		Exponent in 1 Term		Exponents in Both Terms	
		D.		D.		D.
+ times a sum	$2a(b+c) =$	10	$2a(2a+b) =$	12	$2ab(ac+nb) =$	14
	$2a(-b+c) =$	9	$2a(-a+b) =$	12	$2ab(-ac+nb) =$	10
− times a sum	$-2a(b+c) =$	12	$-2a(a+b) =$	13	$-2ab(ac+nb) =$	18
	$-2a(-b+c) =$	12	$-2a(-a+b) =$	12	$-2ab(-ac+nb) =$	24
+ times a Dif.	$2a(b-c) =$	9	$2a(a-b) =$	11	$2ab(ac-nb) =$	10
	$2a(-b-c) =$	10	$2a(-a-b)$	11	$2ab(-ac-nb) =$	15
− times a Dif.	$-2a(b-c) =$	12	$-2a(a-b) =$	11	$-2ab(ac-nb) =$	23
	$-2a(-b-c) =$	11	$-2a(-a-b) =$	9	$-2ab(-ac-nb) =$	25

+ times trinomial, all terms positive 21
− times trinomial, all terms negative 37
+ times trinomial or greater, mixed signs 35
− times trinomial or greater, mixed signs 33

Legend: Difficulty expressed in terms of wrongs per one hundred pupils per one representative problem.

TABLE XI

RELATIVE DIFFICULTY OF LEARNING-UNITS IN SPECIAL PRODUCTS

	Lit. and known				Two literal terms			
					No. Num. Coef.		Num. Coefficients	
		D.		D.		D.		D.
(sum)², no Exp. . . .	$(a+1)^2$	13	$(a+3)^2$	13	$(a+b)^2$	14	$(2a+3b)^2$	22
Expon.	$(a^3+1)^2$	22	$(a^3+3)^2$	24	$(a^3+b)^2$	24	$(2a^3+3b)^2$	34
(dif.)², no Exp.	$(a-1)^2$	19	$(a-3)^2$	22	$(a-b)^2$	20	$(2a-3b)^2$	17
Expon.	$(a^3-1)^2$	30	$(a^3-3)^2$	29	$(a^3-b)^2$	30	$(2a^3-3b)^2$	32
(sum)(dif.)	$(a+1)(a-1)=$			9	$(a+b)(a-b)=$			8
	$(a+3)(a-3)=$			7	$(2a+3b)(2a-3b)=$			14
Exponents	$(a^2+1)(a^2-1)=$			9	$(a^2+b)(a^2-b)=$			10
	$(a^2+3)(a^2-3)=$			11	$(2a^2+3b)(2a^2-3b)=$			*

Legend: Relative difficulty expressed in terms of wrongs per one hundred pupils per one representative problem.

* Inadvertently omitted from the test, therefore no data.

Note: Experimental data shows that putting the terms in reverse order, as $(3+a)^2$, or $(3+a)(3-a)$, raises the difficulty, but not enough such problems were entered in the tests to give complete data for all possible reversals of above units.

TABLE XII

RELATIVE DIFFICULTY OF LEARNING-UNITS IN DIVISION OF KNOWN NUMBERS

	Divid. > Divis.			Divid. = Divis.			Dividend = 0		
	Ex.	V.	H.		V.	H.		V.	H.
$+\div+$	$\frac{+4}{+2}$	1	5	$\frac{+4}{+4}$	8	8	$\frac{0}{+4}$	10	12
$-\div-$	$\frac{-4}{-2}$	5	4	$\frac{-4}{-4}$	7	8	$\frac{0}{-4}$	15	16
$+\div+$	$\frac{+4}{-2}$	3	4	$\frac{+4}{-4}$	8	8			
$-\div+$	$\frac{-4}{+2}$	2	2	$\frac{-4}{+4}$	7	7			

TABLE XII. *Continued.*

Product Divided by Number

Dividend	Divisor Positive	Dif.	Divisor Negative	Dif.
(+)(+)	$\frac{(+4)(+2)}{+2}$	5	$\frac{(+4)(+2)}{-2}$	5
(−)(−)	$\frac{(-4)(-2)}{+2}$	11	$\frac{(-4)(-2)}{-2}$	11
(+)(−)	$\frac{(+4)(-2)}{+2}$	6	$\frac{(+4)(-2)}{-2}$	7

Legend: V., relative difficulty in terms of wrongs per one hundred pupils per one representative problem, vertical form. H., relative difficulty in terms of wrongs per one hundred pupils per one representative problem, horizontal form.

Note: Under division of a product by a number, vertical form only is used, relative difficulty being expressed in terms of wrongs per one hundred pupils per one representative problem.

TABLE XIII

RELATIVE DIFFICULTY OF LEARNING-UNITS IN DIVISION OF MONOMIALS BY MONOMIALS

	$+\div+$	D.	$-\div-$	D.	$+\div-$	D.	$-\div+$	D.
No. Num. Coef.	$\frac{a}{a}$	5	$\frac{-a}{-a}$	10	$\frac{a}{-a}$	10	$\frac{-a}{a}$	12
Num. Coef. in Divd.	$\frac{4a}{a}$	9	$\frac{-4a}{-a}$	13	$\frac{4a}{-a}$	14	$\frac{-4a}{a}$	11
Num. Coef. in both	$\frac{4a}{2a}$	10	$\frac{-4a}{-2a}$	9	$\frac{4a}{-2a}$	8	$\frac{-4a}{2a}$	8
Involving printed exponents	$\frac{4a^3}{2a}$	7	$\frac{-4a^3}{-2a}$	7	$\frac{4a^3}{-2a}$	12	$\frac{-4a^3}{2a}$	5
Divisor a known	$\frac{4a}{2}$	6	$\frac{-4a}{-2}$	8	$\frac{4a}{-2}$	4	$\frac{-4a}{2}$	8

TABLE XIV. Relative Difficulty of Learning-units in Division of Polynomials by Monomials

	Sum ÷ +	D.	Sum ÷ −	D.	Dif. ÷ +	D.	Dif. ÷ −	D.
Divisor a known	$\frac{4a+2}{2}$	22	$\frac{4a+2}{-2}$	24	$\frac{4a-2}{2}$	17	$\frac{4a-2}{-2}$	19
Divisor a literal	$\frac{4a+ab}{a}$	*	$\frac{4a+ab}{-a}$	*	$\frac{4a-ab}{a}$	*	$\frac{4a-ab}{-a}$	*
Ditto with printed exponents......	$\frac{4a^3+2ab}{2a}$	24	$\frac{4a^3+2ab}{-2a}$	35	$\frac{4a^3-2ab}{2a}$	21	$\frac{4a^3-2ab}{-2a}$	29
	$\frac{m^3n^2+2mn}{mn}$	19	$\frac{m^3n^2+2mn}{-mn}$	30	$\frac{m^3n^2-2mn}{mn}$	20	$\frac{m^3n^2-2mn}{-mn}$	37
Trinomial ÷ Mon.	$\frac{4mn+8m^4n^2-2mnk}{2mn}$			40	$\frac{4mn+8m^4n^2-2mnk}{-2mn}$			43

TABLE XV. Relative Difficulty of Learning-units in Division of Product by a Monomial

	+ times a +		− times a −		+ times −, or reverse	
		D.		D.		D.
Divisor +	$\frac{(4a^2b^2)(2ab)}{ab}$	35	$\frac{(-4a^2b^2)(-2ab)}{ab}$	*	$\frac{(4a^2b^2)(-2ab)}{ab}$	52
Divisor −	$\frac{(4a^2b^2)(2ab)}{-ab}$	*	$\frac{(-4a^2b^2)(-2ab)}{-ab}$	40	$\frac{(4a^2b^2)(-2ab)}{-ab}$	52
	$\frac{(a^2b^2)(-2ab)(-4abc)}{\pm ab}$	60				

... per one hundred pupils per one representative problem.

TABLE XVI

RELATIVE DIFFICULTY OF LEARNING-UNITS IN REMOVAL OF PARENTHESES

	Preceded by a + sign				Preceded by a − sign			
(Sum) or (Dif.)		D.		D.		D.		D.
Knowns	$4+(3+5)=$	6	$4+(3-5)=$	11	$4-(3+5)=$	25	$4-(3-5)=$	21
	$-4+(3+5)=$	18	$-4+(3-5)=$	21	$-4-(3+5)=$	29	$-4-(3-5)=$	28
Lit's.	$4a+(3+5a)$	11	$4a+(3-5a)$	14	$4a-(3+5a)$	26	$4a-(3-5a)$	25
	$-4a+(3+5a)$	24	$-4a+(3-5a)$	19	$-4a-(3+5a)$	36	$-4a-(3+5a)$	26
	(Sum) − *(Sum)* $(2a+3b)-(5a+6b)$			27	*(Sum)* − *(Dif.)* $(2a+3b)-(5a-6b)$			26
	(Dif.) − *(Sum)* $(2a-3b)-(5a+6b)$			43	*(Dif.)* − *(Dif.)* $(2a-3b)-(5a-6b)$			22

Double parentheses:

Preceded by + and − . . . . $2a+[4a-(3b-5a)]$ 34
Preceded by − and + ... $2a-[4a+(3b-5a)]$ 40
Preceded by − and − . . . . $2a-[4a-(3b-5a)]$ 36

Legend: Relative difficulty is expressed in terms of wrongs per one hundred pupils per one representative problem.

TABLE XVII

RELATIVE DIFFICULTY OF LEARNING-UNITS IN ALGEBRAIC CONCEPTS

Concepts of terms

	D.		D
Monomial	.51	Literal factor	.53
Binomial	.29	Coefficient	.21
Polynomial	.38	Numerical coefficient	.53
Literal number	.28	Base	.52
Literal term	.33	Power of a number	.66
Factors	.34	Exponent	.32
Common factor	.69	Absolute value	.45

Procedure Concepts	D.
Necessity of prefixing a + sign to a positive number	.44
Necessity of prefixing a − sign to a negative number	3
Sign of the sum when adding like sign numbers	.13
Sign of sum when adding unlike sign numbers	2
Procedure in addition of literal numbers	.48
Procedure in subtraction	4
Manners of indicating multiplication	2
Sign of product of two positive numbers	2
Sign of product of two negative numbers	7
Sign of product of positive and negative numbers	.10
Arrangement of factors in a product	.21
Manipulation of exponents in a product	.28
Sign of quotient from division of like sign numbers	9
Sign of quotient from division of unlike sign numbers	6
Manipulation of exponents in a quotient	.43
Division of a product by a number	.65
Order of operations when a complicated expression involves all four fundamental processes	.34
Placing within parentheses when preceded by a + sign	5
Placing within parentheses when preceded by a − sign	.13
Removal of parentheses when preceded by a + sign	.15
Removal of parentheses when preceded by a − sign	4
Removal of double parentheses	.57

Legend: Relative difficulty expressed in terms of wrongs per one hundred pupils.

ERROR-TYPES

The value of an adequate study of recurrent errors in pupils' test papers is rated highly by all investigators in the algebra field, for as Rugg and Clark [11] state the matter: "It has been shown that success in the teaching of algebra depends primarily on the teacher's knowledge of the typical difficulties which pupils will face in learning algebra."

Rugg and Clark list errors under two general heads: "(1) accidental errors (*i.e.*, errors of reading, writing, following directions, arithmetic errors, and the like), and (2) recurring errors." [12] Their second term seems well chosen, but the first one is entirely

[11] Rugg, H. C., and Clark, J. R., "Standardized Tests and the Improvement of Teaching in First Year Algebra," *School Review*, XXV, p. 115.

[12] Rugg, H. C. and Clark, J. R., *loc. cit.*, p. 205.

too inclusive for a critical study. Their bifurcation is not well taken, as an "accidental" error might also be "recurrent" and therefore need careful analysis and emphasis. It would seem better, for purposes of bifurcation to use the general heads of *Algebraic Errors* and *Arithmetical Errors*, for certainly we are interested in knowing what percent of algebraic difficulty, so called, is in reality arithmetical difficulty. Among the algebraic errors there is possible a clear bifurcation similar to theirs. There are "recurrent" errors which stand out clearly in any study. There are other errors which have a frequency of only one or two in our 350 cases. Again pupils fail to solve many problems where the wrong answer could have been arrived at in two or more different ways. There are also wrong answers which it is absolutely impossible to explain in any apparent way; and, finally, there are omissions in the body of the test where presumably the pupil does not know how to work the problem, and therefore passes it by. All of these types mentioned since "recurrent" errors, are listed as "unclassified." Thus every pedagogical unit will list *Arithmetical Errors*, and a more or less extended list of *Recurrent Errors* which pertain to the working of problems in the particular pedagogical unit.

To appreciate the significance of the findings of this research in the matter of error-types, it is essential that we have some idea of the few previous studies of errors in elementary algebra.

Rugg has conducted the most extended study of errors in algebra.[13] His excellent sampling of pupils taking first year algebra is unquestioned. He used 518 pupils who were *just finishing first year algebra.* The italicized words indicate the possible criticism of the results. In order to identify the errors which hinder pupils in vertical subtraction of monomials, for example, the check-up should be made when they have just completed the study of such a pedagogical unit—not after several months of more or less continuous use of the learning-units involved, with frequent correction of such errors by the teacher, until most of the units are functioning automatically. There is also some ground for criticism in that his sampling of the learning-units is a very narrow one.

Fossler's study [14] cannot be defended as being at all conclusive

[13] Rugg, H. O., "The Experimental Determination of Standards in First Year Algebra," *School Review*, XXIV, pp. 37–66, Jan., 1916.

[14] Fossler, Merritt Luther, *A Study of the Errors Made by Students Who Have Completed First Year Algebra*, Iowa Master's Thesis, 1924.

respecting error-types on the ground of his extremely limited sampling of learning-units. Fossler used four of the Douglas Standard Diagnostic Tests. These tests included (1) addition and subtraction, with 10 representative problems, (2) multiplication, with 10 representative problems, (3) division, with 10 representative problems, (4) simple equations, with 10 problems. So far as comparison with our results is concerned his error-data is drawn from only 30 test problems with a total of 2,254 errors, including those in equations. Our study includes over, 1,200 problems, with an analysis of 43,828 errors. Another criticism lies in the nature of the test material used. The test in addition lists problems such as $\begin{array}{r}7m\\-4m\\\hline\end{array}$ but no problems where the augend is less than the addend. The above problem might be solved correctly on the basis of subtraction of the 4 from the 7, an arithmetic situation; but with the smaller number above the larger, we have a purely algebraic situation, and a much more difficult learning-unit. The same general criticism occurs in connection with the type of subtraction problems where the following occurs $\begin{array}{r}7x,\\4x\\\hline\end{array}$ but no representation of the more difficult unit of $\begin{array}{r}4x.\\7x\\\hline\end{array}$ The first problem might be solved correctly with no comprehension of the change of sign necessary in the subtrahend in algebraic subtraction, whereas the second unarithmetical situation would not be likely to be so solved.

Palmer's study [15] can be criticized on the same grounds as those applied to the Fossler study. Palmer's tests include only 36 problems, and 12 of these were on equations and formulas, items not covered by our study. The 24 problems that parallel our learning-units gave a total of only 1,997 errors for study, compared to our 43,828 errors. Moreover, Palmer lists his errors under skills involved; he does not list the type of error that occurs. Palmer's sampling of pupils is also rather small (165 cases).

The author makes no claim that his data on error "flow" is exact or final. However, with the exception of the unit covering vertical multiplication of known numbers, where the zero-unit

[15] Palmer, Eber Lenon, *The Analysis of Algebra in Relation to the Learning Process*, Iowa Master's Thesis, 1925.

errors total 75 percent of the total errors due to the failure of the texts to present any drill on such units, it is very likely that our data are the most accurate up to the present date, and are very significant respecting types of errors and their relative contribution to the difficulty met by the pupil. Table XXXVII gives an idea of the approximate situation. From this table we note that the outstanding difficulty in connection with the N processes of our study is *Functional*—31 percent of the total errors found. *Sign* difficulty is next in percent of difficulty with a total of 22.9 percent of the total errors recorded. *Exponent* errors total 8.5 percent, and *Carelessness* errors total 8.2 percent of the total.

Comparison with some of the other findings is worthy of notice. Rugg and Clark list sign errors as 50 percent of the difficulty in removal of parentheses—our data show 46.9 percent. They call attention to the fact that errors connected with the removal of parentheses preceded by a plus sign are only about one-third as numerous as errors where the parentheses are preceded by a minus sign. Our data, from equal problem representation in the two cases, show a ratio of 835 to 2,146, slightly under their findings.

With *Special Products* they find that cross multiplication, or sum times a difference, furnish 26.7 percent of the total errors; our data show 26.1 percent. They suggest in a note that the high percent may be due to using the formula applicable to the square of a binomial. Our data show that 14.2 percent of the errors connected with cross multiplication are of such an origin.

Due to the composite nature of the Rugg-Clark tests, it is impossible to carry out this interesting type of comparison throughout the field.

Close comparison with Fossler's error study is difficult. He notes that exponent errors constitute 26.89 percent of the total; our data show 8.5 percent. He finds sign errors to contribute 24.4 percent of total difficulty; our data show sign errors to stand at 22.9 percent. He lists process errors at 23.11 percent; our data show them to make up 31 percent of total difficulty.

Comparison with Palmer's findings are practically impossible. He does note that parentheses preceded by a minus sign give almost five times as many errors as when the parentheses are preceded by a plus sign. This is much larger than either our findings or Rugg's.

TABLE XVIII

RELATIVE FREQUENCY OF ERROR-TYPES CONSTITUTING PUPIL DIFFICULTY IN ALGEBRAIC ADDITION OF KNOWN NUMBERS

Error-types	Example		V.	% D.	H.	% D.
No – sign No. used as Neg. No.	$\begin{array}{r}+9\\4\\\hline+5\end{array}$		54	14.4	12	4.2
No – sign No. ignored		$\begin{array}{r}+9\\4\\\hline+9\end{array}$	10	2.8	1	.3
Subtract rather than add	$\begin{array}{r}-9\\-4\\\hline-5\end{array}$		17	4.9	35	12.0
Prefix opposite sign to sum		$\begin{array}{r}-9\\-4\\\hline+13\end{array}$	23	6.3	28	9.6
Errors of unlike signs						
Arithmetic addition	$\begin{array}{r}-9\\+4\\\hline13\end{array}$		65	17.6	46	15.9
Prefix sign of smaller No. to sum		$\begin{array}{r}-9\\+4\\\hline+5\end{array}$	47	12.8	48	16.4
Zero-unit errors						
Sign number ignored	$\begin{array}{r}0\\+4\\\hline0\end{array}$		14	3.8	5	1.7
Change sign in sum		$\begin{array}{r}-4\\0\\\hline+4\end{array}$	5	1.4	6	2.1
Unclassified algebraic errors			70	18.0	56	19.3
Arithmetic errors			70	18.0	54	18.5
				100.0		100.0
Summary of errors:						
Sign errors				59.		60.4
Unclassified algebraic errors				18.0		19.3
Arithmetic errors				23.0		20.3
				100.0		100.0

Legend: Columns headed respectively V. and H. give the relative prevalence of error-types expressed in terms of errors per one hundred pupils in tests using vertical and horizontal forms of problems; columns headed %, indicate contribution of each error-type to the total pupil-difficulty in vertical and horizontal forms of addition known numbers, % being determined on basis of total raw data.

TABLE XIX

RELATIVE FREQUENCY OF ERROR-TYPES CONSTITUTING PUPIL-DIFFICULTY IN ALGEBRAIC ADDITION OF LITERAL NUMBERS

Error-types	Example	V.	% D.	H.	% D.
Errors of like sign numbers					
No − sign No. used as Neg. No.	$\begin{array}{r}4a\\+9a\\\hline+5a\end{array}$	48	8.3	28	6.2
No − sign number ignored	$\begin{array}{r}4a\\+9a\\\hline+9a\end{array}$	29	4.9	10	2.2
Subtract rather than add	$\begin{array}{r}-9a\\-4a\\\hline-5a\end{array}$	72	12.0	51	10.8
Prefix opposite sign to sum	$\begin{array}{r}-9a\\-4a\\\hline+13a\end{array}$	24	4.0	44	9.4
Errors of unlike sign numbers					
Arithmetic addition	$\begin{array}{r}-9a\\4a\\\hline13a\end{array}$	41	6.8	47	10.1
Prefix sign of smaller No.	$\begin{array}{r}-9a\\+4a\\\hline+5a\end{array}$	29	4.9	47	10.1
Literal number errors					
Ignore No. with no Num. Coef.	$\begin{array}{r}a\\a\\\hline a\end{array}$	24	3.9	3	.6
	$\begin{array}{r}-a\\-4a\\\hline-4a\end{array}$	93	15.5	24	5.1
Omit literal No. from sum	$\begin{array}{r}-9a\\4a\\\hline-5\end{array}$	44	7.3	15	3.2
Change column to horizontal	$\begin{array}{r}-9a\\4a\\\hline-9a+4a\end{array}$	6	1.0		
Errors resulting from use of quantity $(a-b)$ as monomial	$\begin{array}{r}(a-b)\\-4(a-b)\\\hline-3a-b\end{array}$	43	7.0	11	2.2
Zero-unit errors					
Sign number ignored	$\begin{array}{r}0\\-4a\\\hline0\end{array}$	5	.8	5	1.1
Change sign of the sum	$\begin{array}{r}0\\-4a\\\hline+4a\end{array}$	2	.2	8	1.6
Unclassified algebraic errors		112	18.7	146	31.0
Arithmetic errors		28	4.7	31	6.4
			100.0		100.0

Summary of errors in Add'n of Lit. No.'s.

Sign errors	41.0	50.0
Lit. No. errors	33.7	11.4
Unclassified Alg. errors	18.7	31.0
Arithmetic errors	5.5	7.6
Column errors	1.1	
	100.0	100.0

Legend: Columns headed V. and H. give respectively the relative prevalence of error-types expressed in terms of errors per one hundred pupils in tests using vertical and horizontal forms of problems. Columns headed %, indicate contribution of each error-type to the total pupil-difficulty in vertical and horizontal forms of addition of literal numbers, % being determined on basis of total raw data.

TABLE XX

RELATIVE FREQUENCY OF ERROR-TYPES CONSTITUTING PUPIL-DIFFICULTY IN ADDITION OF POLYNOMIALS

Error-types	Frequency	% D.
Subtract instead of add	21	6.1
Add like − signs, give sum opposite sign	34	9.9
Arithmetic addition of like signs	59	17.1
Add unlike − signs, give sum sign of smaller No.	41	11.8
$a + a = a$	29	8.5
Ignore literal No. without Num. Coef.	8	2.3
Omission of literal number in sum	14	3.9
Errors in copying	50	14.4
Omission of term from the sum	23	6.6
Unclassified algebraic errors	30	9.4
Arithmetic errors	35	10.0
		100.0

SUMMARY OF ERROR-TYPES IN ADDITION OF POLYNOMIALS

Sign errors	45.0
Literal number errors	14.6
Transcribing errors	21.0
Unclassified algebraic errors	9.4
Arithmetic errors	10.0
	100.0

TABLE XXI

SUMMARY OF ERROR-TYPES CONSTITUTING TOTAL DIFFICULTY IN ALGEBRAIC ADDITION

	% D.
Sign errors	49.6
Literal number errors	14.8
Column-errors	.3
Transcribing errors	3.4
Unclassified algebraic errors	20.2
Arithmetic errors	11.7
	100.0

Legend: Frequency expressed in terms of errors per one hundred pupils. % of contribution of respective error-types to total pupil-difficulty is determined on the basis of total raw data.

TABLE XXII

RELATIVE FREQUENCY OF ERROR-TYPES CONSTITUTING PUPIL-DIFFICULTY IN ALGEBRAIC SUBTRACTION OF KNOWN NUMBERS, VERTICAL AND HORIZONTAL FORMS

Error-types	Example	V.[66]	% D.	H.	% D.
Like signs					
Add and change sign of sum	+9 +4 −13	14	4.8	22	2.8
Arithmetic difference	+4 +9 +5	3	1.0		
Prefix sign of smaller No.	+9 +4 −5	11	3.8	4	.4
Sign of process taken for sign of the subtrahend	5 − (3) = +8			176	22.1
Unlike signs					
Add, prefix sign of smaller No.	+9 −4 −5	11	3.9	18	2.2
Change sign of minuend	+9 −4 −13	60	21.2	156	19.5
Algebraic addition	+9 −4 +5	104	36.6	309	39.7
Algebraic Add'n with 0	0 +4 +4	47	16.7	46	5.7
Unclassified Alg. errors		34	12.2	61	6.7
Arithmetic errors		21		19	

Summary of error-types for subtraction of known numbers:

Sign errors	21.3
Process errors	59.6
Errors with 0	8.0
Unclassified	8.1
Arithmetic	3.0
	100.0

Legend: Columns headed respectively V. and H. give the relative prevalence of error-types expressed in errors per one hundred pupils for vertical and horizontal forms. Columns headed % D., indicate contribution of each error-type to the total pupil-difficulty in the form indicated. In the summary it is total difficulty in subtraction of known numbers.

TABLE XXIII

RELATIVE FREQUENCY OF ERROR-TYPES CONSTITUTING PUPIL-DIFFICULTY IN ALGEBRAIC SUBTRACTION OF LITERAL NUMBERS, VERTICAL AND HORIZONTAL FORMS

Error-types	Example	V.	% D.	H.	% D.
Add, change sign of sum	$+5a$ $+3a$ $-8a$	30	5.0	53	4.2
Add, sign of smaller No.	$+5a$ $-3a$ $-2a$	36	6.0	57	4.5
Algebraic addition	$+5a$ $-3a$ $+2a$	199	33.6	730	57.9
Change sign of minuend	$+5a$ $-3a$ $-8a$	119	20.4	156	12.4
Ignore Lit. No. without Num. Coef.	$+5a$ a $+5a$	67	11.3	42	3.3
Omit literal number in Ans.	$+5a$ $+2a$ $+3$	1.1	1.1	21	1.6
Errors in use of $(a-b)$				67	5.3
Algebraic Add'n with 0	0 $5a$ $5a$	73	12.3	23	1.7
Zero given value of 1	$5a-0=+4a$			14	1.1
Zero minus No. = zero	$0-(+5)=0$			16	
Unclassified Alg. errors		61	10.3	102	8.0
Arithmetic errors		7		20	

Summary of error-types for subtraction of literal numbers:

Sign errors	28.1
Process errors	43.8
Errors with 0	4.4
Lit. No. errors	11.3
Unclassified	9.6
Arithmetic errors	2.8
	100.0

Legend: Columns headed respectively V. and H. give the relative prevalence of error-types expressed in errors per one hundred pupils for vertical and horizontal forms. Columns headed % D., indicate contribution of each error-type to the total pupil-difficulty in subtraction of literal numbers, with the form indicated. In summary, % is in terms of total difficulty in subtraction of literal numbers.

TABLE XXIV

RELATIVE FREQUENCY OF ERROR-TYPES CONSTITUTING PUPIL-DIFFICULTY IN INVERTED SUBTRACTION OF KNOWN NUMBERS

	Frequency	% Dif.
Add, and change the sign of the sum	51	18.0
Change the sign of the minuend	158	55.9
Add instead of subtract	32	11.4
Zero less a number equals zero	14	5.1
Use of $+0$ or -0	6	2.2
Unclassified algebraic errors	18	7.4
Arithmetic errors	10	
		100.0

TABLE XXV

RELATIVE FREQUENCY OF ERROR-TYPES CONSTITUTING PUPIL-DIFFICULTY IN ALGEBRAIC SUBTRACTION OF POLYNOMIALS

	Frequency	% Dif.
Like sign errors	18	3.1
Unlike sign errors	33	5.6
Algebraic addition instead of subtraction	233	38.9
Change sign of minuend and add	153	25.6
Omission of literal number in answer	8	1.2
Errors in copying	53	8.9
Errors in omission or arrangement of terms	21	3.5
Unclassified algebraic errors	79	13.2
Arithmetic errors	45	
		100.0

TABLE XXVI

GRAND SUMMARY OF RELATIVE FREQUENCY OF ERROR-TYPES CONSTITUTING TOTAL PUPIL-DIFFICULTY IN ALGEBRAIC SUBTRACTION

	% of Total Difficulty
Sign errors	25.4
Process errors	50.2
Errors with use of zero	5.8
Literal number errors	6.7
Unclassified Alg. errors	9.0
Arithmetic errors	2.9
	100.0

Legend: Percent of difficulty computed from total raw data.

TABLE XXVII

RELATIVE FREQUENCY OF ERROR-TYPES CONSTITUTING PUPIL-DIFFICULTY IN VERTICAL AND HORIZONTAL FORMS OF MULTIPLICATION OF KNOWN NUMBERS

Error-type	Example	V.	% D.	H.	% D.
Like signs = a Neg.	−4 −2 —— −8	10	8.3	80	29.9
Unlike signs = a Pos.	−4 +2 —— +8	7	5.3	62	23.0
Zero unit errors	−4 0 —— −4	95	75.7	74	27.7
Add instead of multiply	−4 +2 —— −2	9	7.4	12	4.4
Subtract instead of multiply	−4 +2 —— −6			1	.3
Unclassified Alg. errors		1	.7	19	7.0
Arithmetic errors		3	2.6	21	7.7
			100.0		100.0

Legend: Columns headed respectively V. and H. give the relative prevalence of error-types expressed in terms of errors per one hundred pupils in vertical and horizontal forms of problems. Columns headed % D., indicate contribution of each error-type to the total pupil-difficulty in multiplication of known numbers with the form indicated; % determined on basis of total raw data.

TABLE XXVIII

RELATIVE FREQUENCY OF ERROR-TYPES CONSTITUTING PUPIL-DIFFICULTY IN MULTIPLICATION OF MONOMIALS

Error-type	Example	Frequency	% of Total Difficulty
Like signs = a Negative	$(-2a)(-4b) = -8ab$	76	7.9
Unlike signs = a Pos.	$(-2a)(+4b) = +8ab$	91	15.6
Add numerical Coef's	$(-2a)(+4b) = +2ab$	34	3.6
Subtract Num. Coef's	$(-2a)(+4b) = -6ab$	6	.6
Omit Num. Coefficients	$(-2a)(+4b) = -ab$	8	.8
Multiply exponents	$(+2a^2)(-4a^3) = -8a^6$	6	.6
Subtract exponents	$(+2a^2)(-4a^5) = -8a^3$	7	.7
Omit Exp. in product	$(+2a^2)(-4a^3) = -8a$	5	.5
Ignore unprinted Exp.	$(+2a)(-4a^3) = -8a^3$	371	38.6
Lit. No. Exp. applied to Num. Coeffcient	$(+2a)(-4a^2) = -32a^3$	31	3.1
Lit. No. omitted in Pro.	$(+2a)(-4a) = -8$	82	8.6
Fail to combine Lit. No.'s	$(+2a)(-4ab) = -8aab$	53	10.4
Unclassified Alg. errors		118	7.4
Arithmetic errors		16	1.6
			100.0

Legend: Frequency is expressed in terms of errors per one hundred pupils. % of total difficulty is determined on the basis of total raw data.

TABLE XXIX

RELATIVE FREQUENCY OF ERROR-TYPES CONSTITUTING PUPIL-DIFFICULTY IN THE MULTIPLICATION OF POLYNOMIALS BY MONOMIALS

Error-type	V.	H.	% D.
Like signs = a negative	50	45	6.4
Unlike signs = a positive	71	73	9.1
Ignore unprinted exponent	117	101	13.9
Subtraction of exponents	31	6	2.5
Multiplication of exponents	21	12	2.2
Omit exponents	76	19	6.3
Add or Subt. Num. Coefficients	11	5	1.0
Ignore numerical coefficients	38	16	3.5
Omit numerical coefficients	28	42	4.3
Omit literal number	158	141	19.0
Fail to combine literal numbers	18	12	2.9
Confuse literal numbers	19	12	2.0
Omit a term	35	29	4.1
Combine unlike terms	17	98	7.0
Unclassified algebraic errors	140	84	14.5
Arithmetic errors	12	10	1.3
			100.0

Legend: Columns headed respectively V. and H. give relative prevalence of error-types expressed in terms of errors per one hundred pupils in vertical and horizontal forms of problems. Column headed % D. indicated the contribution of each error-type to the total pupil-difficulty in multiplication of polynomials by monomials, percentage being based on raw data of both forms, combined.

TABLE XXX

RELATIVE FREQUENCY OF ERROR-TYPES CONSTITUTING PUPIL-DIFFICULTY IN THE MULTIPLICATION OF SPECIAL PRODUCTS

Sum or Difference Squared

Error-type	Freq.		% D.
(Sum)^2, one term minus $(a+b)^2 = a^2 + 2ab - b^2$	8		
(Dif.)^2, second term plus $(a-b)^2 = a^2+2ab+b^2$	19		
(Dif.)^2, third term minus $(a-b)^2 = a^2-2ab-b^2$	40	Sgn. Er.	14.8
Fail to multiply second product term by 2	87		20.0
Add numerical coefficients in second term	14		3.2
Mult. numerical coefficient by exponent	10		2.2
Omit a letter or term in product	49		11.4
Omit an exponent in product	33		
Use of a wrong exponent	73		
Multiply instead of add exponents	5	Exp. Er.	25.6
Use of sum times difference formula	23		5.3
Unclassified algebraic errors	61		14.0
Arithmetic errors	15		3.5
			100.0

Sum Times Difference or Reverse

Fail to square one term $(a+b)(a-b)=a^2-b$	7		4.5
Fail to square Num. Coef. $(2a+b)(2a-b)=2a^2-b^2$	9		6.0
Use of wrong exponent	22		
Omit exponents in product $(a+b)(a-b)=a-b$	29	Exp. Er.	33.2
Use wrong formula $(a+b)(a-b)=a^2+2ab+b^2$	22		14.2
Second term positive $(a+b)(a-b)=a^2+b^2$	25		16.6
Unclassified algebraic errors	34		22.0
Arithmetic errors	5		3.5
			100.0

Legend: Frequency expressed in terms of errors per one hundred pupils. % of total difficulty of each error type computed from the total raw data.

TABLE XXXI

SUMMARY OF RELATIVE FREQUENCY OF ERROR-TYPES CONSTITUTING TOTAL PUPIL-DIFFICULTY IN ALGEBRAIC MULTIPLICATION

Type of Error	Total No.	% of Dif.
Sign errors	2,310	19.8
Zero unit errors	586	5.0
Add instead of multiply	291	2.5
Subtract instead of multiply	22	.2
Omit numbers or terms	2,050	17.6
Exponent errors	3,235	27.8
Combination of terms errors	945	8.1
Use of wrong formula errors	580	5.0
Unclassified algebraic errors	1,366	11.7
Arithmetic errors	272	2.3
	11,657	100.0

TABLE XXXII

RELATIVE FREQUENCY OF ERROR-TYPES CONSTITUTING PUPIL-DIFFICULTY IN ALGEBRAIC DIVISION OF KNOWN NUMBERS, VERTICAL AND HORIZONTAL FORMS

Error-types	Example	V.	% D.	H.	% D.
Like signs = negative	$\frac{-4}{-2}=-2$	51	18.8	11	13.9
Unlike signs = positive	$\frac{-4}{+2}=+2$	66	24.2	10	13.3
No. ÷ No. = No.; or 0	$\frac{-4}{-4}=-4$ or 0	37	13.6	18	23.4
Zero unit errors	$\frac{0}{2}=2$ or $+0$ or -0	46	17.0	24	21.6
Unclassified Alg. Errors		29	10.6	5	6.8
Arithmetic errors		42	15.8	9	12.0
			100.0		100.0

Legend: Columns headed respectively V. and H. give relative prevalence of error-types expressed in terms of errors per one hundred pupils for vertical and horizontal forms of problems. % of difficulty for each error-type is computed from total raw data.

TABLE XXXIII

RELATIVE FREQUENCY OF ERROR-TYPES CONSTITUTING PUPIL-DIFFICULTY IN DIVISION OF MONOMIALS BY MONOMIALS

Error-type	Example	Fre-quency	% of Dif.
Like signs = a negative	$\frac{-4a}{-2a}=-2$	29	8.0
Unlike signs = positive	$\frac{-4a}{2a}=2$	82	22.5
Add or subtract	$\frac{-4a}{-2a}=-6a$	18	4.9
Fail to divide Num. Coef.	$\frac{4a^2}{2a}=4a$	7	1.9
Lit. No. errors	$\frac{a}{a}=a \quad \frac{a}{a}=0 \quad \frac{a^2}{a}=1^2$	133	36.4
Omit literal number	$\frac{4a^2}{2a}=2$	17	4.7
Add exponents	$\frac{4a^2}{2a}=2a^3$	12	3.2
Other unclassified Exp. errors		23	6.3
Unclassified algebraic errors		42	11.5
Arithmetic errors		2	.6
			100.0

Legend: Frequency is expressed in terms of errors per one hundred pupils. % of difficulty is computed from the total raw data.

TABLE XXXIV

RELATIVE FREQUENCY OF ERROR-TYPES CONSTITUTING PUPIL-DIFFICULTY IN DIVISION OF POLYNOMIALS BY MONOMIALS

Error-type	Example	Frequency	% of Dif.
Like signs = negative	(as in Table xxiii)	90	6.5
Unlike signs = positive	ditto	256	18.3
Add or subtract	ditto	61	4.4
Fail to divide Num. Coef.	ditto	29	2.0
Literal No. errors	ditto	285	20.4
Omit literal number	ditto	39	2.8
Add exponents	ditto	31	2.3
Other exponent errors		77	5.5
Div. only one term of sum or dif.	$\frac{2a-2}{2} = 2a-1$	174	12.5
Div. both terms of product	$\frac{(4a)(6b)}{2} = 6ab$	172	12.2
Omit term in quotient		10	.7
Combine unlike terms	$\frac{4ab+6ac}{2} = 6abc$	23	1.7
Unclassified Alg. errors		139	10.0
Arithmetic errrors		10	.7
			100.0

Legend: Frequency is expressed in terms of errors per one hundred pupils. % of difficulty is computed from total raw data.

TABLE XXXV

SUMMARY OF RELATIVE FREQUENCY OF ERROR-TYPES CONSTITUTING TOTAL PUPIL-DIFFICULTY IN ALGEBRAIC DIVISION

Error-type	Total No.	% of Dif.
Sign errors	1,971	28.1
Add or subtract	260	3.7
Fail to divide numerical coefficient	117	1.7
Zero unit errors	327	3.4
Number divided by itself = self or 0	1,561	22.3
Omit number or term in quotient	219	3.1
Exponent errors	471	6.7
Combination of terms errors	78	1.1
Divide only one term of sum or difference	572	8.1
Divide both terms of a product	557	8.0
Unclassified algebraic errors	752	10.7
Arithmetic errors	217	3.1
	7,012	100.0

TABLE XXXVI

RELATIVE FREQUENCY OF ERROR-TYPES CONSTITUTING PUPIL-DIFFICULTY IN THE REMOVAL OF PARENTHESES

Error-type	Example	Frequency	% Dif.
Change sign when preceded by a plus sign	$2a+(-3a)=2a+3a=+5a$	80	
Fail to change sign preceded by a minus sign	$2a-(-3a)=2a-3a=-a$	383	
Change sign of term outside of parentheses	$(2a+b)-3a=2a+b+3a=5a+b$	40 Sign Err.	46.9
Errors with double parentheses		59	5.5
Failure to combine like terms		129	
Combination of unlike terms		70	18.6
Errors in algebraic addition		90	8.4
Errors in algebraic subtraction		123	11.4
Errors in copying		16	1.5
Unclassified algebraic errors		60	5.6
Arithmetic errors		22	2.1
			100.0

Legend: Frequency expressed in terms of errors per one hundred pupils. % of difficulty computed from total raw data.

TABLE XXXVII

SUMMARY TABLE OF GENERAL ERROR-TYPES CONSTITUTING TOTAL PUPIL-DIFFICULTY IN *N* PROCESSES IN FIRST YEAR ALGEBRA

Type of Error	Frequency	% T. Dif.
Sign errors	10,046	22.9
Process errors*	13,611	31.0
Literal number errors†	2,414	5.5
Zero unit errors	1,790	4.1
Exponent errors	3,706	8.5
Combination of terms errors‡	1,633	3.7
Carelessness errors§	3,546	8.2
Unclassified algebraic errors	5,019	11.4
Arithmetic errors	2,063	4.7
	43,828	100.0

* Contains such types of error as using the wrong process, such as: addition when instructed to subtract; use of wrong formula in special products; failing to change sign of subtrahend in subtraction, etc.

† Contains such error-types as $a+a=a$, $a+a=0$, $2a-a=2$, etc.

‡ Contains such error-types as failure to combine like terms, and combination of unlike terms.

§ Contains such error-types as omission of letters or terms, errors in copying terms, etc.

TABLE XXXVIII

SHOWING THE RELIABILITY OF TESTS, NUMBER OF LEARNING-UNITS, AND NUMBER OF ERROR-TYPES FOUND FOR EACH PEDAGOGICAL DIVISION OF THE *N* PROCESSES COVERED BY THE RESEARCH

Pedagogical Division	Test R.	No. L.-u.	No. Er.-Tp.'s
Vert. Add'n knowns	.91	25	10
Hor. Add'n of knowns	.89	25	10
Vert. Add'n of literals	.91	31	15
Hor. Add'n of literals	.93	31	14
Addition of polynomials	.83	13	11
Vert. Subt'n of knowns	.94 *	19	9
Hor. Subt'n of knowns	.94 *	20	9
Vert. Subt'n of literals	.93	32	9
Hor. Subt'n of literals	.95	35	13
Inverted subtraction	.91	12	7
Subt'n of polynomials	.93	12	9
Vert. Multiplication knowns	.89 *	24	6
Hor. Multiplication knowns	.89 *	30	7
Multiplication of monomials	.90	48	14
Monomial times polynomial	.91	28	16
Spec. Products $(\text{Binom.})^2$	.87 *	16	13
Spec. Products, Sum times Dif.	.87 *	8	8
Vert. Division of knowns	.92 *	16	6
Hor. division of knowns	.92 *	16	6
Monomials divided by Monom.	.89 *	20	10
Polynomials divided by Monom. (including product ÷ Monom.)	.89 *	25	14
Removal of parentheses	.93	19	11
Algebraic concepts	.69	36	

Legend: Reliability of tests determined by correlation between halves, and use of Brown's formula. Second column, number of learning-units in pedagogical division. Third column, number of error-types found in each pedagogical division of the field; in each case inclusive of unclassified algebraic errors, and arithmetic errors.

* In case of two identical neighboring reliabilities, the two types were included in one text, and the reliability stated is for the test as a whole.

TEXTBOOK ANALYSIS

A study of the textbook analysis sheet (Table XXXIX) cannot but impress one with the fact that there is little or no agreement among textbook writers as to the amount of teaching, or the number and distribution of drill problems. This could not well be otherwise without an adequate analysis of the learning-units in each field, and some knowledge of the relative difficulty which they present to the pupil.

TABLE XXXIX

TEXTBOOK ANALYSIS SHOWING CM. OF COLUMN LENGTH OF TEACHING MATERIAL, AND NUMBER OF DRILL PROBLEMS OFFERED BY EACH OF THE THREE TEXTS

Pedagogical Division	New.-Harper		Schor.-Clark		Wells-Hart	
	Cm.T.	D.P.	Cm.T.	D.P.	Cm.T.	D.P.
Vert. Add'n of knowns		13		88		18
Hor. Add'n of knowns	52.0 *	10	55.5 *	0	33.0 *	36
Vert. Add'n literals		8		11		12
Horiz. Add'n literals	3.5 *	7	1.5 *	4	12.0 *	15
Addition of polynomials	11.5	21	16.0	24	18.0	21
Total for addition	67.0	59	72.8	127	63.0	102
Vert. Subt'n of knowns		8		35		8
Horiz. Subt'n of knowns		31		4		2
Inverted Subt'n		10		0		45
Total for Subt'n of knowns	20.0	49	16.0	39	13.0	55
Vert. Subt'n of literals	.0	5	.0	35	.0	16
Horiz. Subt'n of literals	.0	5	.0	1	.0	9
Subt'n. of polynomials	2.0	10	2.0	8	7.5	26
Total for subtraction	22.0	69	18.0	83	20.5	106
Vert. multiplication of knowns		3		0		0
Hor. multiplication of knowns	9.5 *	8	13.0 *	149	21.0 *	85
Multiplication of monomials	23.0	17	16.0	30	16.0	58
Mon. times polynom., Vert. Form		12		1		4
Mon. times polynom., Hor. Form	15.0 *	26	12.0 *	27	15.0 *	15
Spec. products, Binom. squared		30		43		38
Spec. Prod., sum times Dif.	23.0 *	18	20.0 *	31	14.0 *	39
Total for multiplication	70.5	116	61.0	282	66.0	239
Division of knowns	4.0	6	4.0	28	23.0	43
Div. of monomials by monomials	5.0	28	9.0	31	15.0	43
Div. of Polynom. by monomials	4.0	32	5.0	10	7.0	26
Total for division	13.0	66	18.0	69	45.0	112
Removal of Parenth. Precd. by +	0.0	0	0.0	0	0.0	7
Removal of Parenth. Precd. by −	1.0	0	1.0	0	5.0	38
Placing in Parenth., etc.	6.0	60	2.5	23	23.0	40
Total for parentheses	7.0	60	3.5	23	28.0	85
Algebraic concepts	71.0		52.0		101.0	
Grand total for N processes	250.5	370	225.3	584	323.5	644

Legend: First column under each textbook name shows vertical centimeters of teaching material, second column shows offering in drill problems.

* Summary for division indicated and the blank preceding this one.

The value of r between drill problems offered by each text and the wrongs per one hundred pupils is $+.13$; between drill problem offerings and errors per one hundred pupils the value of r is $+.08$. These figures would seem to indicate that the number of drill problems offered has no effect upon learning. Common sense tells us that this ground is impossible. We are forced to the conclusion, therefore, that the teacher equation, and the possible failure to hold the class drill to the plan outlined by the research, have invalidated the results we hoped to gain respecting the results of the varying amounts of drill provided by the three textbooks. These findings indicate the necessity of more carefully controlled experimentation than was possible in this research.

Conclusions

A serious student of the data herein presented cannot fail to note several significant facts. Among these are: (1) the extreme complexity of the mental functions involved in first year algebra. It must be kept in mind, however, that what appears on analysis to be a more or less hopeless conglomeration of interrelated complexities, will become for the pupil under proper amounts of drill, a smoothly operating chain of automatically functioning psychological links—links which have been forged one by one as learning has progressed from the simple to the complex processes. (2) Although algebra has been taught and learned without an adequate published analysis of the learning-units available (and probably not even available in the minds of the authors or teachers since the literature repeatedly points out the need of such an analysis, but fails to provide for the need [15]) this research has proved the possibility of making such a study, and indicated the line of further research in order to gain a scientific analysis of the total field. (3) Only in the light of such an analysis of learning-units can pedagogically adequate tests be constructed, drill material appraised, and the "flow" of error frequency properly determined.

In the light of this research, it is perhaps permissible for the writer to indulge in some observations which point the way to possible improvement of the general situation. (1) In as much

[15] One such analysis for the subtraction of one integral monomial from another similar one has been given. See Smith, D. E. and Reeve, W. D., *The Teaching of Junior High School Mathematics*, pp. 200–201. Ginn & Co., 1927.

as large numbers of the teachers of algebra will not have, for many years to come, a thorough preparation and training for their task, it is greatly to be desired that textbooks and drill books be constructed in such a scientific manner as to give representation to all learning-units which the pupil must master in order to have an adequate basis for later mathematical situations which he may be called upon to meet. (2) It is very essential that prospective algebra teachers have in all cases a thorough understanding of the types of errors which pupils will be likely to make in the working of problems incidental to each pedagogical unit of the field of first year algebra. (3) It is also highly desirable that they be forewarned as to the relative difficulty of the learning-units in each division of the field, so that they may properly apportion their class periods to the various types of problems which occur.

A STUDY OF PUPIL REACTIONS

By LULU E. McWILLIAMS

So much has been written on the number and the types of questions which teachers ask or should ask, that one is in danger of considering only the technique involved and of forgetting the fundamental objective of questioning as a form of teaching. The stimulation of real thought should be a definite aim. Such stimulation is indicated by thought contributions from the pupils' individual experiences and by questions of the thought-provoking kind. For if children are thinking, really trying to solve the problem at issue, they will have questions of their own. Monroe says, "the frequency and logical significance of the questions asked by the pupils supply one of the very best measures of the efficiency of a class conference." An occasional survey of one's teaching from the standpoint of the pupils' reactions, therefore can be of great value to the teacher as a check and guide. Moreover, the presence of intellectual interests—which are always to be aroused and developed in effective teaching—is often indicated by the questions and pupil contributions made during the class period.

It was with these facts in mind that the present study was made—a study of the kinds of questions and the frequency of other individual contributions, and the factors influencing either or both.

Since the investigation was somewhat limited by the time allowed, it was thought best to make the observations in one course only, that the results might be more comparable. The course chosen was the one in eighth grade mathematics given in the University of Chicago High School under the title of "Mathematics 2B." The class was made up of two sections. In the one section there were twenty-four pupils all of whom were either eighth or ninth graders; the other section was composed of twenty-four pupils also, but two of this number were tenth graders while all of the rest were straight eighth graders. All of the members in both sections had had the regular course in unified mathematics as given in the seventh grade except one

girl who entered from a South American school during the latter part of her seventh school year. So the mathematical background of the pupils could be assumed on the whole to be fairly even and for that reason the questions had all the more weight.

The observations were made over a period of about four weeks, data being secured on eighteen class periods.

The method followed was quite simple. Four things were noted and recorded each day—the number of simple questions asked by the pupils, the number of thought questions and contributions, the type of work occupying the major part of the class period, and the general class attention.

By a "simple question" is meant that type of question that calls for the "yes" or "no" answer, or for working information, or perhaps for a repetition or further explanation of information already given. Illustrations of the simple question are as follows: "Does this come out even?", "Have I done this right?", "How do you know how to move the decimal point?", "How can you get the equation for these?", "Aren't those the same ones we worked yesterday?" and "How many problems are we going to have?"

"Thought question" is used to mean that kind of question which shows that the pupil has been doing some thinking on his own part and is seeking more information or is putting a new problem before the class.

So closely connected with thought questions are "individual contributions" that they are not listed separately here. They are those definite original suggestions which tend to show easier or different methods of solution of the problem at hand. Very few such contributions were given by either section but a good example of those given is: "In 50 minutes a man rows 10 miles downstream and in 1 hr. and 30 min. he rows 12 miles upstream. Find the rate of the current and how fast he can row in still water." The entire class worked it in the following way:

$$\begin{aligned} x+y&=12 \\ x-y&=\ 8 \\ \hline x&=10 \\ y&=\ 2 \end{aligned}$$

but one boy was thinking hard and suggested the following solution explaining why he took each step:

$$x + y = x - y + 4,$$
$$x = 10,$$
$$y = 2.$$

Another illustration of individual contribution is:

"A does a piece of work in 5 days and A and B working together do it in 2 days. How long does it take B to do it if he works alone?"

The class got the equation $1/5 + 1/x = 1/2$ or $1/x = 3/10$. One boy asked if $x/1 = 10/3$ was the same thing. This certainly showed that he was doing some original thinking along the line of the given problem.

We shall here list a few of the "thought questions" asked in order that the reader may the more clearly understand the nature and significance of our work. "Why wouldn't it be easier to combine $5x/100$ and $8x/100$ first and then solve the equation?" "Just what is meant by 'exceed'?" "Why isn't this right? It proves." "You can't have a negative answer, can you for then the problem would mean nothing?" "May I work the tenth problem decimally? It would be much more simple." "Will you tell me wherein I'm wrong? I didn't let x equal what Mr. S did."

The work of the class period was classified into four parts: *presentation, assimilation, organization*, and *recitation*. By far the greater part of the observations were on assimilation, either of the supervised study or discussion type. The general topic for each day as indicated by the class exercise was also noted. Since the efforts of the observer were concentrated on hearing and classifying pupil reactions, it was impossible to make detailed studies of class attention. Therefore general estimates only were made and recorded as "fair," "good," and "excellent."

The data showed a wide variance from day to day. Simple questions varied from one to twenty-eight; thought questions and individual contributions from two to fifteen (Tables I and II). Moreover, the two stood in no definite relation to each other, at times the simple questions were much in the majority, at other times the two approached the same number, and on still other days thought questions and contributions took the lead. The reasons for these differences were sought in the data at hand.

TABLE I

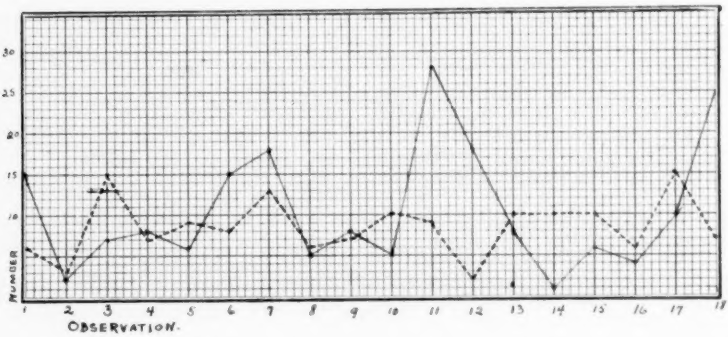
FINDINGS

Observation	Type of Lesson	Topic	Number of Simple Questions	Number of Thought Questions	Attention
1	Presentation and Assimilation	Interest	15	6	Fair
2	Assimilation	Interest	2	3	Good
3	Assimilation and Presentation	Interest	7	15	Excellent
4	Assimilation	Review	8	7	Excellent
5	Assimilation	Interest	6	9	Good
6	Assimilation	Review	15	8	Fair
7	Assimilation	Percentage	18	13	Good
8	Assimilation and Presentation	Simple Problems in Two Unknowns	5	6	Excellent
9	Assimilation and Presentation	Simple Problems in Two Unknowns	8	7	Good
10	Assimilation	Percentage	5	10	Excellent
11	Assimilation	Percentage	28	9	Fair
12	Assimilation	Interest	18	2	Fair
13	Assimilation and Presentation	Interest	8	10	Fair
14	Discussion, Drill and Assimilation	Solution of Equations	1	10	Excellent
15	Drill	Solution of Equations	6	10	Excellent
16	Drill	Solution of Equations	4	6	Excellent
17	Assimilation	Equations and Percentage	10	15	Good
18	Drill	Solution of Equations	25	7	Fair

The most striking differences (a variation of from sixteen to nineteen points in favor of simple questions) were noted in observations 11, 12, and 18 (Table II). They may possibly be explained by the following facts. The first period was a continuation of somewhat mechanical work of the previous period and interest was lagging. For this reason there was much confusion, which meant that much material had to be repeated. The second was a period of supervised study and individual conferences; in such a period there was always great danger that the observer would not catch the gist of what was being said. Often the questions were asked in very low tones; this is most desirable

TABLE II

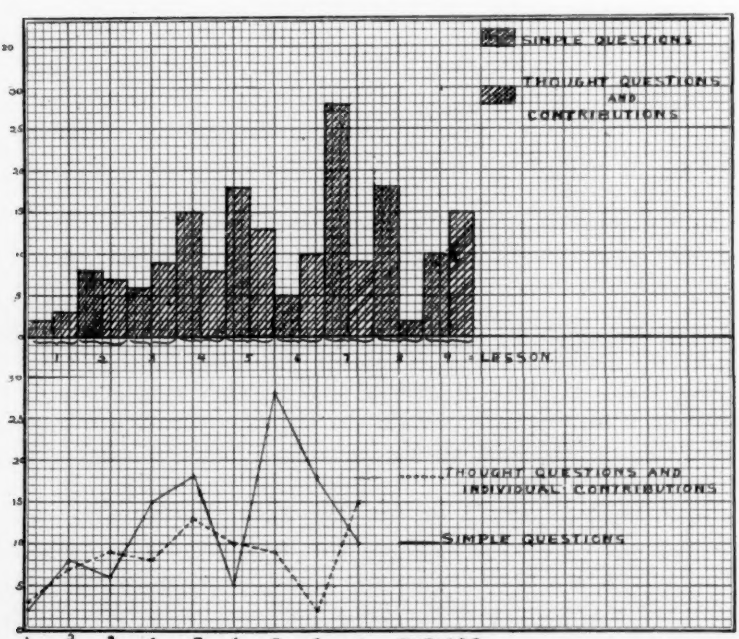
Graphs Showing Relation between Simple Questions and Thought Contributions



———Simple Questions
. . . . Thought Questions and Individual Contributions

TABLE III

Assimilation Periods

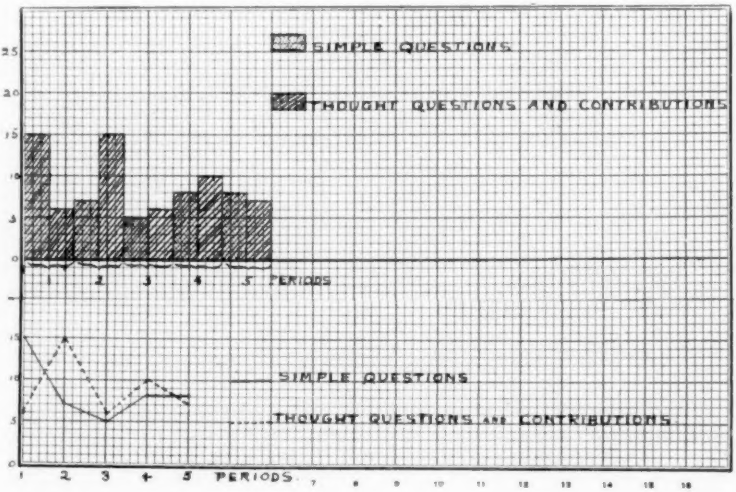


in a study period, but it undoubtedly accounts for apparent discrepancies in the chart of assimilation periods. (Chart III.). The third was a drill lesson. It seemed to the observer that the children were hardly ready for drill and this necessitated an abundance of needlessly foolish questions. If the instructor had spent five minutes, when he first observed that the pupils did not get the gist of the situation, to make a more extended explanation than he had formerly made, many of these questions would never have been asked.

In four instances the thought questions and simple questions were only one point apart. In each of these four cases the lesson was of an assimilation type and the questions few in number. This showed that the lesson had been well presented, that the pupils were interested in mastering the situation, and that assimilation was really taking place.

TABLE IV

PRESENTATION AND ASSIMILATION PERIODS

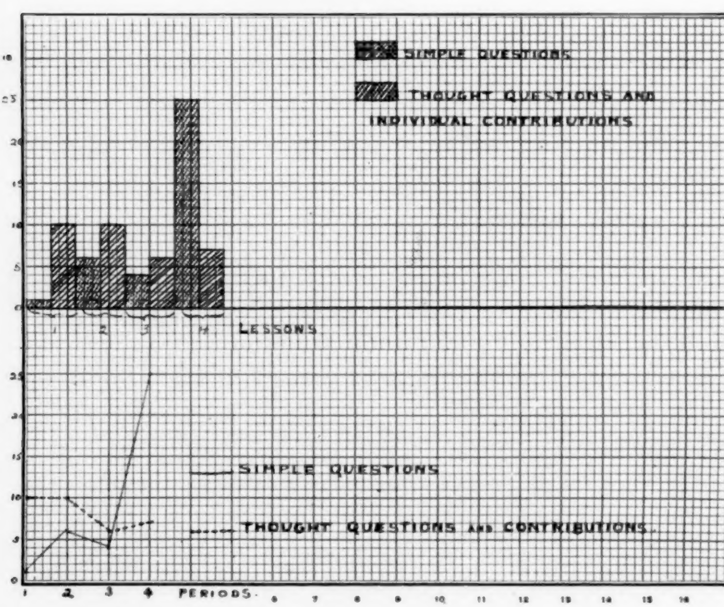


There were a number of cases, however, in which the thought contributions outnumbered the simple questions from one to nine points inclusive—observations 2, 3, 9, 8, 10, 13, 14, 15, 16, and 17. This showed that the topic had been well presented, had excited interest, and that the pupils were on the road to learning.

The above study suggested that there might be a more real relation between simple and thought material in those periods which were devoted entirely to assimilation and to those given over to presentation and assimilation. So Charts III and IV were made. It was found that in Lessons 1, 2, and 3 in Table III and in Lessons 3, 4, 5 in Table IV that the questions of the two types balanced each other but it was not found true in the other cases. In three or one third of the assimilation periods the thought material proved to be equal in number to one half or more of the simple questions and in two cases the thought contributions were one and a half times as numerous as the simple questions. Table IV shows that in one case the simple questions doubled the thought questions, and in another case that the opposite is true, while in the three remaining lessons the questions balanced. Table V shows the relationship between the two kinds

TABLE V

DRILL PERIODS



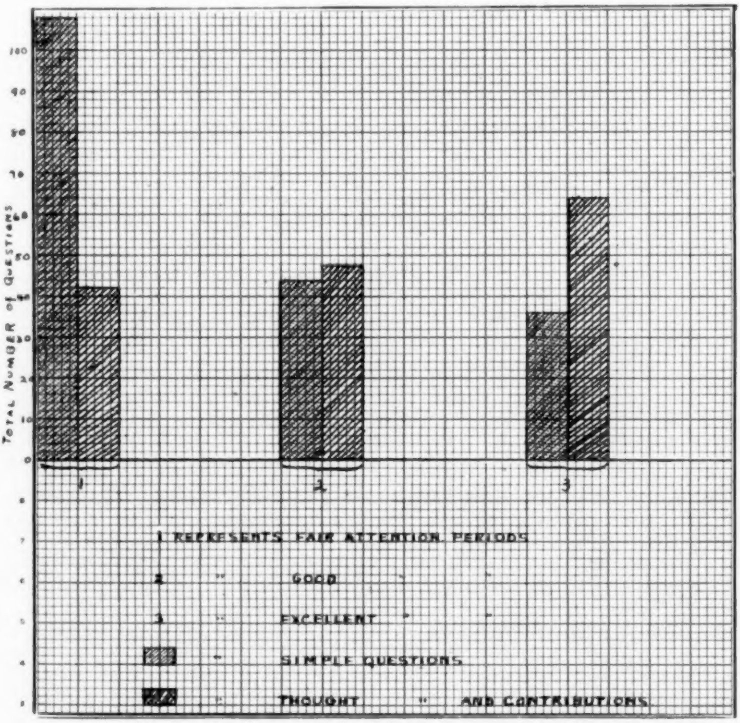
of questions in drill lessons to be in favor of thought questions. But there was a class period which showed that there were two

simple questions asked for every thought question. This was due to the fact that the children were not in the working mood and the instructor did not get them there until the class period was nearly over (Table V).

The relation of simple questions and thought material to class attention was quite evident (Table VI). The former, as might

TABLE VI

RELATION EXISTING BETWEEN ATTENTION AND QUESTIONS



be expected, decreased markedly with increased attention. The latter, however, showed no such marked change.

To attempt to draw final conclusions from such data as this paper presents would be a mistake. In the first place the time allowed for the study was entirely too short for convincing and conclusive work; secondly the allowances which must be made for errors in such a short study are too great. The difficulty of

securing data in supervised study periods makes records of less value. Yet limited as they may be, it does seem that the findings have meaning; they are indicative of the present situation, and suggestive of causes and of methods of attack.

This report will be of value to show teachers that diminishing simple questions rests with them. The teacher's task then is to provide a rich mass of assimilative material, to follow up thought questions by class discussion, to lessen simple questions by holding the student's attention, to give clear-cut presentations and explanations, and to check up on themselves occasionally.

THE VALIDATION OF NEW TYPES OF PLANE GEOMETRY TESTS

By HARRY A. GREENE AND RUTH O. LANE

University of Iowa

Teachers in all fields of subject matter are turning to objective tests as the only reliable means of determining the efficiency of their instruction. This interest on the part of the classroom teacher has had a most wholesome effect on the recent developments in educational measurements. As teachers themselves become better informed they grow more critical of existing devices, and in many cases have themselves made significant contributions. In fact, a very considerable portion of the material described herein is the product of a classroom teacher working under normal instructional conditions.

Unit-Achievement Tests.—Until quite recently teachers and others interested in the development and use of educational tests have been satisfied to use general survey tests of achievement. Now they are convinced that the greatest good for themselves as well as for their pupils comes from the more frequent use of smaller unit tests (not necessarily brief tests) designed to furnish valid and reliable measures of each major unit of instructional material as soon as instruction on each unit is completed. Among the practical advantages arising from such a practice are the following:

1. The test units themselves may be made to more accurately parallel the content of the course of study upon which instruction is based. This results in greater assurance of *validity* in the tests.
2. The individual test units may be made somewhat longer, thus permitting a more extensive sampling of the important skills or items of information, which in turn increases the *reliability* of the resulting measures.
3. The periodical use of unit-achievement tests, based on practical units of instructional material, insures the teacher *an opportunity to identify the specific difficulties of*

> *individual pupils early enough in the course to give corrective instruction at a time when it will count. End-of-the-year testing obviously does not do this.*

The new types of tests presented and described in this article are the result of an effort to go beyond this earlier conception of educational tests as mere survey instruments, to the more advanced notion of tests as specific guides to teachers in adjusting instruction to the needs of their individual pupils. It is the outgrowth of over five years of experimental work in the development of objective testing in geometry carried on in the University High School by Miss Ruth Lane, the junior author of this article.

These tests are designed to secure for the instructor reliable and economical measures of the information and the skills which contribute to successful achievement in plane geometry. They are prepared [1] in two equal and parallel forms covering the basic units of instructional material in the subject. There are six tests in all, three of which are to be given during the first semester and three during the second semester. Each test is standardized for use immediately upon the completion of instruction on the unit of subject matter which it tests. Very often this will come at the end of the month or six weeks period, but whether or not this is the case, the teacher will invariably wish to give a test at the completion of each of these basic units.

Description of the Tests.—With the exception of the first test each test unit is divided into four parts. Parts 3 and 4 of each of the remaining tests place a heavy emphasis upon the proving of theorems and experience with practical applications of geometrical principles in exercises. Test 1 covers the preliminary work before formal theorems are taken up. It consists of a series of 34 exercises and questions designed to measure comprehension of the vocabulary and the basic definitions of geometry. It may be given to the class in from two to four weeks after the beginning of the instruction in the subject.

The subject matter usually presented under the general heading of Parallel Lines and Triangles is covered in Test 2. The test is so constructed that it does not matter whether the triangle is studied before or after parallel lines. Part 1 of this test

[1] These tests, known as the *Lane-Green Unit-Achievement Tests in Plane Geometry*, are published by Ginn and Company, Boston, 1929.

introduces what is believed to be a novel method of testing the ability of the student to indicate *what is to be proved* in a theorem. This part of the test consists of a series of important theorems and problems in which certain words and phrases are bracketed together and numbered. The student is instructed to read each exercise carefully and write on the line at the right the numbers of the bracketed portions which contain the statement of what is to be proved (or what is given) in the theorem or problem. This is illustrated as Type 4 on page 00 of this article where the samples of the several kinds of objective exercises used in these tests are given. Part 2 of this test measures in a similar way the student's ability to indicate what is given in the theorem. Both of these skills undoubtedly contribute significantly to the pupil's ability to comprehend the complete theorem, and thus successfully to undertake its proof (Part 3). Part 4 consists of 27 miscellaneous exercises based upon parallel lines and triangles.

Test 3 is similar in organization to that followed in Test 2, except that, of course, the subject matter deals with *rectilinear figures,* including exercises on parallelograms and polygons in general in addition to exercises on parallel lines and triangles.

Test 4 includes the topic of locus in addition to the usual material taught concerning *the circle.* Locus is usually begun before the study of the circle is begun, although this practice is not entirely uniform. On this account no questions on loci are included in Test 3. It is a common practice for the subject of locus to receive very little instructional emphasis following the completion of the work on the circle, although in a very few cases it follows the treatment of the circle as a separate unit. However, if the subject of locus is not taken up sufficiently by the time the work on the circle is completed the teacher may give the section at a later time.

The work with *similar polygons* involves an understanding of proportion. Accordingly, the first part of Test 5 deals with certain simple concepts of ratio and proportion as found in ordinary arithmetical and algebraic situations. The remainder of the test follows the same general organization as that used in Test 4.

Test 6 covers the general principles which are used in the calculation of the lengths of lines and of the areas of polygons.

The first part deals with lengths of lines; the second part with the comparison of areas; and the third and fourth parts with the proving of theorems and the solving of miscellaneous problems dealing with areas of polygons.

Order of Use of Tests.—It is probable that in the classroom use of the tests the first three tests will be given in practically the order in which they are numbered. Tests 4, 5, and 6 need not be given in order. In fact, any order may be used for these three tests without loss, with one exception. If the topic of *Similar Polygons* (Test 5) is taught before *The Circle* (Test 4) there will be four questions in Test 5 which the pupil will not be prepared to answer. Since this order of presentation is rarely used it will affect only a very few schools. In any case in which *Similar Polygons* is taught prior to *The Circle* the teacher may mark the exercises in Test 5 which are affected and instruct the pupils not to attempt them. In view of the fact that the time limits are fairly exacting the pupil will not actually lose as much as four points, for he will have proportionately more time to work on other exercises.

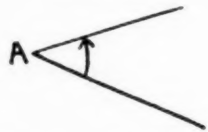
Types of Exercises Used.—The nature of the subject matter of plane geometry makes it somewhat difficult to develop valid exercises which are at the same time thoroughly objective. For example, the testing of the student's ability to prove a theorem or to go through the steps in a construction problem involves many factors which are difficult to objectify. In these tests a special effort has been made to test objectively those very difficult phases of geometry. Three common types of objective exercises are utilized, namely; (1) straight recall exercises, (2) completion exercises involving one or two words, (3) multiple response exercises of four or five choices. In addition to these common forms of test exercises three newer and somewhat novel types are introduced. For convenience in identifying them in this discussion these forms are called (4) [2] bracketed phrases, (5) order of procedure, (6) wrong, useless and out of order statements in proofs. Each of these types of exercises is illustrated below.

[2] These numbers also serve to identify the type of exercise among the illustrations given herewith.

TYPE 1. STRAIGHT RECALL

a. How many geometric surfaces has a chalk box? 6

b. What word describes point *A* on this angle? vertex



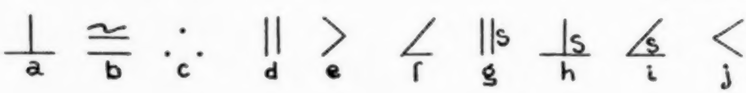
TYPE 2. COMPLETION

a. The whole is equal to the sum of all its parts

TYPE 3. MULTIPLE RESPONSE

a. The number of straight lines which may be drawn through one point is: (*a*) none; (*b*) one; (*c*) two; (*d*) three; (*e*) an indefinite number. e

b. The symbol for perpendicular is the one marked by the letter a



TYPE 4. BRACKETED PHRASES

Indicating What is to be Proved

Two triangles (1) are congruent (2) if three sides of one triangle are equal respectively (3) to three sides of the other (4). 1, 2

Indicating What is Given

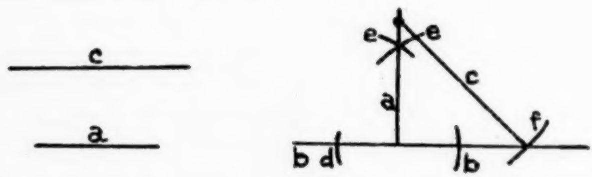
The areas of two triangles are to each other (1) as the squares of any two corresponding sides (2), if the triangles (3) are similar (4). 3, 4

TYPE 5. ORDER OF PROCEDURE

A right triangle has been constructed using the given line *c* as the hypotenuse and the given line *a*

as a side. Write the letters *a*, *b*, *c*, *d*, *e*, *f*, so that they will show the order in which the lines they represent were drawn. bdeafe

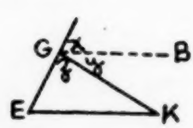
(When the same letter is used to mark two lines it means that these two constructions are interchangeable in order.)



TYPE 6. WRONG, USELESS, OR OUT OF ORDER STATEMENTS

a. The sum of the angles of a triangle is 180°. ***Which of the following statements in the proof of this theorem is wrong, useless, or out of order?*** (The reasons are omitted.) b

a. $x = E$
b. $x = y$
c. $y = K$
d. $x + y + z = 180°$
e. $G + K + E = 180°$



Special care was taken in the formulation of the multiple response exercises used. Such exercises are valuable only when all of the responses represent equally likely answers to the problem to the uninformed student. Many times five-response exercises are actually nothing more than alternate response exercises to students who know anything at all about them and are not guessing. This is because the other two or three responses are so obviously not possible answers that the correct answer is arrived at by *exclusion* rather than as a result of a definite, *positive knowledge.* With a very few exceptions the multiple-response exercises used in these tests are made up from the five responses most often given by large numbers of students during the early stages of this project. More than forty-six thousand responses of students to exercises of this type were analyzed and tabulated as the basis for these exercises.

In the case of the construction exercises (illustrated by Type 5 above) the factor of chance is carefully controlled. In giving

the letters *a, b, c, d, e, f,* in the order in which the construction each represents was made, the pupil has only *1* chance in over *700* of guessing the correct order, if there is but one answer which is accepted as correct. In the selection of *What is Given* and *What is to be Proved* in Parts 1 and 2 of Tests 2 and 3 the chance of guessing the right order is *1* out of *15*. In most objective examinations the operation of chance is in theory at least probably much greater than this.

Special Features of the Tests.—An examination of existing types of objective tests in plane geometry reveals a number of features which to the authors seem to be significant inadequacies. In the first place, it seems to be one of the generally accepted purposes of instruction in plane geometry to bring the student to the point where he can follow through the steps of reflective thinking, and determine whether or not a valid conclusion has been reached. In this respect most objective tests fail, for in order to secure the necessary objectivity the exercises utilized consist mainly in matters of information, or of superficial memory of more or less unrelated details. In the second place, many objective test exercises in plane geometry avoid almost entirely the type of situation in which the student is called upon to *think through the entire proof of a theorem.* Exercises which accomplish this are usually lacking in objectivity.

The plan of testing the student's ability to prove theorems in these tests is in some respects unique. The same may be said of the method by which constructions are tested. The sample exercise, Type 6 above, illustrates the method of testing the student's ability to prove theorems. In this type of exercise we are testing *indirectly* the pupil's ability to follow through the proof of the theorem, and *directly* the ability to "spot" errors and irrelevancies—skills he is called upon to use constantly both in and out of the geometry classroom. The pupil is rightly called upon to correct his own and other's work in class. The ability to detect wrong, useless, and out-of-order elements in reasoning is highly related to successful work in this field.[3]

In setting out to test constructions objectively two alternatives are open to the makers of a test. If the actual constructions are to be made the grading of the work can not be absolutely objective. Also within the limits of the time required to take

[3] However, at this moment no actual correlations supporting this statement are available.

the test, fewer exercises can be given because of the added time required to make constructions. In these tests it was decided to sacrifice whatever is lost by not having constructions actually made, in order to gain in objectivity and economy of time. However, the method used is such that the student not only sees *what has been constructed,* but he must also see step by step *how the construction has been made, and the order in which the steps took place.* There is a vast difference between these two procedures.

Validating the Tests; Theorem Content.—There are several possible techniques for validating test material. One of the common ways is to express the validity in terms of the extent to which the tests set up situations calling into play the use of *skills or abilities which experienced and qualified observers consider fundamental to success in the given field.* The recommendations of committees and the opinions of experts in these fields constitute such judgments. In the validation of these tests of plane geometry the major dependence is placed upon the recommendations of the *National Committee on the Reorganization of Mathematics.* The report of this committee is taken as authority for the content of the theorems. *All theorems listed by the Committee in its prescribed list as well as in its subsidiary list are tested directly or indirectly in these tests.* Many of these are tested in more than one way. Every exercise or question in all of the tests dealing with theorems as such or problems requiring the use of theorems can be answered correctly by any student having a knowledge of the theorems listed by the National Committee. This is also true of the constructions which are included in the tests.

Vocabulary Content.—The vocabulary content of the exercises has been carefully guarded. *Only that mathematical vocabulary which is common to eighteen widely used geometry texts is included.* No word which the National Committee advises against using has been introduced in the test exercises. In case a word needed in an exercise is not uniformly used in *all* of the eighteen texts, an explanation is inserted the first time it appears, as for example, *congruent* in Exercise 5, of Part 3, Test 2. Definitions not uniformly agreed upon by the authors of the texts used as the criterion were avoided in the exercises. Questions covering these definitions are either not asked at all,

or are stated in such a way that the disputed point does not enter.

Order of Topics.—The order of presentation of topics is determined by common usage as indicated by the study of eighteen widely used texts. By grouping the subject matter into large units (six well-balanced tests for the entire year, rather than the regular division by "books") much of the difficulty arising from the order of presentation of the topics in the different textbooks is eliminated. In case of doubt as to where a topic belongs it is put in a later test. When the unit is large enough (e.g., locus problems and construction problems) to represent a separate section of a test, the matter of sequence is of less importance, since a section of the test can easily be postponed until after the topic is taught.

TABLE 1

RELIABILITY COEFFICIENTS AND PROBABLE ERRORS OF SCORES 10TH GRADE CLASSES IN PLANE GEOMETRY; 200 CASES PER TEST AND FORM

Test	r_{11}	Form A Standard Deviation	P.E.$_{score}$	r_{11}	Form B Standard Deviation	P.E.$_{score}$
1	.83	5.78	1.61	.81	5.99	1.76
2	.87	9.18	2.23	.89	9.84	2.20
3	.90	8.70	1.85	.90	8.94	1.91
4	.83	6.40	1.78	.78	6.34	2.00
5	.86	7.36	1.85	.83	7.04	1.96
6	.83	6.80	1.89	.86	6.58	1.66

Reliability of the Tests.—The reliability of these tests was determined by the chance-half method of correlation and the later calculation of the reliability coefficients by means of the Spearman-Brown formula. These coefficients together with the P.E.'s of Scores for each form of the six tests are reported in Table 1. On the whole, these coefficients of reliability indicated that the scores on each of the separate tests in this series may be accepted with a large measure of confidence that they represent approximately true measures of the pupil's ability to respond to geometry situations of the type represented by these test exercises.

HUMAN INTEREST PUT INTO MATHEMATICS

By STEPHEN LEACOCK

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The student of arithmetic who has successfully striven with money sums and fractions, finds himself confronted by an unbroken expanse of questions known as problems. These are short stories of adventure and industry with the end omitted, and though betraying a strong family resemblance, are not without a certain element of romance.

The characters in the plot of a problem are three people called *A, B,* and *C*. The form of the question is generally of this sort: "*A, B,* and *C* do a certain piece of work. *A* can do as much work in an hour as *B* in two, or *C* in four. Find how long they work at it."

Or this: "*A, B,* and *C* are employed to dig a ditch. *A* can dig as much in one hour as *B* can dig in two, and *B* can dig twice as fast as *C*. Find how long, etc., etc."

Or after this wise: "*A* lays a wager that he can walk faster than *B* or *C*. *A* can walk half as fast again as *B,* and *C* is only an indifferent walker. Find how far, and so forth."

The occupations of *A, B,* and *C* are many and varied. In the older arithmetics they contented themselves with doing "a certain piece of work." This statement of the case, however, was found too sly and mysterious, or possibly lacking in romantic charm. It became the fashion to define the job more clearly and to set them at walking matches, ditch-digging, regattas, and piling cord wood.

At times they became commercial and entered into partnership, having with their old mystery a "certain capital." Above all they revel in motion. When they tire of walking matches—*A* rides on horseback, or borrows a bicycle and competes with his weakerminded associates on foot. Now they race on locomotives; now on automobile; now they row; or again they become historical and engage stage-coaches; or at times they are aquatic and swim.

If their occupation is actual work, they prefer to pump water into cisterns, two of which leak through holes in the bottom and one of which is water-tight. *A,* of course, has the good one; he also takes the bicycle and the best locomotive, and the right of swimming with the current. Whatever they do they put money on it, being all three sports. *A* always wins.

In the early chapters of the arithmetic, their identity is concealed under the names John, William, and Henry, and they wrangle over the division of marbles. In algebra they are often called *X*, *Y*, *Z*. But these are only their Christian names, and they are really the same people.

Now, to one who has followed the history of these men through countless pages of problems, watched them in their leisure hours dallying with cord wood, and seen their panting sides heave in the full frenzy of filling a cistern with a leak in it, they become something more than mere symbols. They appear as creatures of flesh and blood, living men with their own passions, ambitions and aspirations like the rest of us.

Let us view them in turn. *A* is a full-blooded blustering fellow, of energetic temperament, hot-headed and strong-willed. It is he who proposes everything, challenges *B* to work, makes the bets, and bends the others to his will.

He is a man of great physical strength and phenomenal endurance. He has been known to walk forty-eight hours at a stretch, and to pump ninety-six. His life is arduous and full of peril. A mistake in the working of a sum may keep him working a fortnight without sleep. A repeating decimal in the answer might work him to death.

B is a quiet easy-going fellow, afraid of *A* and bullied by him, but very gentle and brotherly to little *C*, the weakling. He is quite in *A*'s power, having lost all his money in bets.

Poor *C* is an undersized, frail man, with a plaintive face. Constant walking, digging, and pumping have broken his health and ruined his nervous system. His joyless life has driven him to drink and smoke more than is good for him, and his hand often shakes as he digs ditches. He has not the strength to work as the others can, in fact, as Hamlin Smith's arithmetic has said, "*A* can do more work in one hour than *C* in four."

The first time that I ever saw these men was one evening after a regatta. They had all been rowing in it, and it transpired that *A* could row as much in one hour as *B* in two, and *C* in four. *B* and *C* had come in dead fagged and *C* was coughing badly.

Just then *A* came blustering in and shouted, "I say, you fellows, Hamlin Smith has shown me three cisterns in his garden and he says we can pump them until tomorrow night. I bet I can beat you both. Come on. You can pump in your rowing things, you know. Your cistern leaks a little, *C*."

I heard *B* growl that it was a rotten shame and that *C* was

used up now, but they went, and presently I could tell by the sound of the water that *A* was pumping four times as fast as *C*.

For years after that I used to see them constantly about town, and always busy. I never heard of any of them eating or sleeping. Then owing to a long absence from home, I lost sight of them. On my return I was surprised to no longer find *A*, *B*, and *C* at their accustomed tasks; on inquiry I heard that work in this line was now done by *N*, *M*, and *O*, and that some people were employing for algebraical jobs three foreigners called Alpha, Beta, and Gamma.

Now it chanced one day that I stumbled upon old *D*, an aged laboring man who used occasionally to be called upon to help *A*, *B*, and *C*. From this garrulous old man I learned the melancholy end of my former acquaintances. Soon after I left town, he told me, *C* had been taken ill. It seems that *A* and *B* had been rowing on the river for a wager, and *C* had been running on the bank and then sat in a draught. Of course the bank had refused the draught and *C* was taken ill.

A and *B* came home and found *C* dying helpless in bed. *A* shook him roughly and said, "Get up, *C*, we're going to pile wood." *C* looked so worn and pitiful that *B* said, "Look here, *A*, I won't stand this, he isn't fit to pile wood tonight."

C smiled feebly and said, "Perhaps I might pile a little if I sat up in bed." Then *B*, thoroughly alarmed, said, "See here, *A*, I'm going to fetch a doctor; he's caving in." I think that even *A* was affected at the last as he stood with bowed head, aimlessly offering to bet with the doctor on *C's* labored breathing.

"*A*," whispered *C*, "I think I'm going fast." "How fast do you think you'll go, old man?" murmured *A*.

"I don't know," said *C*, "but I'm going at any rate." *C* rallied for a moment and asked for a certain piece of work that he had left downstairs. *A* put it in his arms and he expired.

B burst into a passionate flood of tears and sobbed, "Put away his little cistern and the rowing clothes he used to wear. I feel as if I could never dig again."

The funeral was plain and unostentatious, except that out of deference to sporting men and mathematicians, *A* engaged two hearses. Both vehicles started at the same time, *B* driving the one containing the remains of his ill-fated friend. *A* on the empty hearse generously consented to a handicap of a hundred yards, but arrived first at the cemetery by driving four times as fast as *B*. (Find the distance to the cemetery.)

NEWS NOTES

The Mathematics Section of the Schoolmen's Week at the University of Minnesota rendered the following program on Thursday, March 28, at 2:00 P.M. in the Old Physics Auditorium, Mr. E. A. Beito, Mechanics Art School, St. Paul, presiding:

"The Contract Plan in Ninth Grade Mathematics," Miss Hannah Nutter, Central High School, Minneapolis.

"The Mathematical Requirements in Commercial Positions," Mr. L. B. Kinney, John High School, St. Paul.

"Some Objectives to be Realized in Plane Geometry," Sister Alice Irene, St. Catherine's College, St. Paul.

"Mathematics in the Service of the Biological Sciences," Mr. J. Arthur Harris, University of Minnesota.

The mid-winter meeting of the Association of Teachers of Mathematics in New England was held on March 9th in the Central High School at Springfield, Mass. The following program was given:

10:30 A.M.

"Opportunities for Broadening the Scope of Junior High School Mathematics," Miss Olive A. Kee, Teachers' College, Boston.

"Algebra in the Making," Elmer R. Bowker, Public Latin School, Boston.

2:00 P.M.

"Methods of Factoring Large Numbers," William Fitch Cheeney, Jr., Tufts College.

"Locophobia, Causes and Cure," George H. Selleck, Phillips Exeter Academy.

The Southeastern Section of the Mathematical Association of America held its seventh annual meeting at Wesleyan College, Macon Georgia, Friday–Saturday, April 19–20.

Wesleyan College gave a special dinner in honor of Professor E. P. Lane, of the University of Chicago.

PROGRAM

Friday, April 19.

7:00 P.M. Special Dinner in honor of Professor E. P. Lane in dining hall of Wesleyan College. Toastmaster, Professor A. B. Morton, Georgia Tech. Addresses by President W. F. Quillian, Wesleyan College, and Professor E. P. Lane, University of Chicago.

9:00 P.M. Social Hour.

Saturday April 20. A. B. Morton, Georgia Tech, presiding.

10:00 A.M. Business Meeting.

10:30 A.M. "Projective Equivalents of Simple Plane Curves," Professor E. P. Lane, University of Chicago; "Functions of Iteration," Professor David F. Barrow, University of Georgia; "Green's Functions Satisfying Non-homogeneous Boundary Conditions," Professor W. W. Elliott, Duke University; "Note on Functions Defined by Limits," Professor W. W. Elliott, Duke University; "The Post-Heroic Age of Geometry," Professor E. P. Lane, University of Chicago; (subject to be announced at meeting), Profes-

sor H. A. Robinson, Agnes Scott College.

W. W. RANKIN, *Sec.*,
Chairman Program Committee, Duke University, Durham, N. C.

The Mathematics Department of Ensley High School in Birmingham, Alabama, with a corps of 12 teachers now has 100 per cent. membership in the National Council of Teachers of Mathematics, according to H. G. Douglas, the head of the department. Let's keep the good work going.

According to Miss Mabel Stewart, of Central High School, Oklahoma City, Oklahoma, their Mathematics Department now has 100 per cent. membership in the National Council of Teachers of Mathematics.

NEW BOOKS

Mathematics in Liberal Education. By FLORIAN CAJORI. Boston: The Christopher Publishing House, 1928. Pp. 169. $1.50.

Modern Algebra. First Course. By RALEIGH SCHORLING AND JOHN R. CLARK. Yonkers: World Book Company, 1929. Pp. 386.

Modern Algebra. Second Course. By RALEIGH SCHORLING, JOHN R. CLARK AND SELMA LINDELL. Yonkers: World Book Company, 1929. Pp. 464.

Essentials of Trigonometry. By DAVID EUGENE SMITH, WILLIAM DAVID REESE AND EDWARD LONGWORTH MORSS. Boston: Ginn and Company, 1928. Pp. 250. $1.44.

New Analytic Geometry. By PERCY F. SMITH, ARTHUR SULLIVAN GALE, AND JOHN HAVEN NEELLEY. Boston: Ginn and Company, 1928. Pp. 326. $2.00.

Exercises and Tests in Junior High School Mathematics. Part I. By DAVID EUGENE SMITH, WILLIAM DAVID REESE, AND EDWARD LONGWORTH MORSS. Boston: Ginn and Company, 1928. Pp. 128. $0.48.

Exercises and Tests in Junior High School Mathematics. Part II. By DAVID EUGENE SMITH, WILLIAM DAVID REESE, AND EDWARD LONGWORTH MORSS. Boston: Ginn and Company, 1928. Pp. 144. $0.48.

Significant Changes and Trends in The Teaching of Mathematics Throughout the World Since 1910. Fourth Yearbook of the National Council of Teachers of Mathematics. New York: Bureau of Publications, Teachers College, Columbia University, 1929. Pp. 186. $1.75.

The Fundamental Skills of Algebra. By JOHN P. EVERETT. New York: Bureau of Publications, Teachers College, Columbia University, 1928. Pp. 109. $1.50.

Enriched Teaching of Mathematics in the High School. By MAXIE WOODRING AND VERA SANFORD. New York: Bureau of Publications, Teachers College, Columbia University, 1928. Pp. 128. $1.50.

Elementary Algebra Problems Made Easy. By ALMA M. FABRICIUS AND WM. HANCE. New York: Progressive Review Book Company, 1929. Pp. 72.

First Steps in Teaching Numbers. By JOHN R. CLARK, A. S. OTIS AND CAROLINE HATTON. Yonkers: World Book Company, 1929. Pp. 225.

Scientific Amusements. By TOM TIT. New York: Thos. Nelson and Sons, 1928. Pp. 414.

Introductory Mathematics. By J. E. ROWE. New York: Prentice-Hall, 1927. Pp. 285. $2.50.

Senior Mathematics, Book I. By ERNST R. BRESLICH. Chicago: University of Chicago Press, 1928. Pp. 335.

Senior Mathematics. Book II. By ERNST R. BRESLICH. Chicago: University of Chicago Press, 1928. Pp. 296.

Trigonometry. By ERNST R. BRESLICH AND CHAS. A. STONE. Chicago: University of Chicago Press, 1928. Pp. 122.

Strayer-Upton Arithmetics. 3 Vols. For Lower, Middle and Upper Grades. By GEO. D. STRAYER AND C. B. UPTON. New York: American Book Company, 1928. Pp. 314, 346, and 394.

Analytic Mechanics. By J. B. REYNOLDS. New York: Prentice-Hall, Inc., 1929. Pp. 347. $4.00.

A History of Mathematical Notations. Vol. I. By FLORIAN CAJORI. Chicago: Open Court Publishing Company, 1928. Pp. 451. $6.00.

A History of Mathematical Notations. Vol. II. By FLORIAN CAJORI. Chicago: Open Court Publishing Company, 1929. Pp. 367. $6.00.

Spherical Harmonics. By T. M. MACROBERT. New York: E. P. Dutton and Company, 1927. Pp. 302. $4.50.

Contributions to Education. By J. CARLETON BELL AND AMBROSE L. SUHRIE. Yonkers: World Book Company, 1928. Pp. 425.

College Algebra. By N. R. WILSON AND L. A. H. WARREN. New York: Oxford University Press, 1928. Pp. 451. $2.08.

Analytic Geometry. By T. E. MASON AND C. T. HAZARD. Boston: Ginn and Company, 1927. Pp. 221. $2.40.

The Triangle Arithmetics. Books I, II and III. By LEO. J. BRUECKNER, C. J. ANDERSON, AND G. O. BANTING. Philadelphia: John C. Winston Company, 1928. Pp. 425, 422, and 469.

Descriptive Geometry. By H. H. JORDAN AND F. M. PORTER. Boston: Ginn and Company, 1929. $3.00.

Plane Trigonometry. By N. J. LENNES AND A. S. MERRIL. New York: Harper and Brothers, 1928. Pp. 179.

Algebra to the Quadratic. By R. W. M. GIBBS. London: Oxford University Press, 1927. Pp. 141.

College Algebra. By KENNETH P. WILLIAMS. Boston: Ginn and Company, 1928. Pp. 310. $2.00.

Mathematics for Students of Technology. By L. B. BENNY. London: Oxford University Press, 1927. Pp. 451. $3.50.

Short Table of Integrals. By B. O. PEIRCE. Boston: Ginn and Company, 1929. Pp. 155. $1.48.

Applied Arithmetic for Girls. By NETTIE STEWART DAVIS. Milwaukee: The Bruce Publishing Company, 1928. Pp. 126. $0.88.

Ninth Grade Mathematics. By FLORA M. DUNN, EMMY S. HUEBNER, AND JOHN S. GOLDTHWAITE. Boston: Ginn and Company, 1929. Pp. 290. $1.20.